Basic Calculation

Formulae

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$
$$(a+b)(a-b) = a^2 - b^2$$
$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$
$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Use the distribution law to show these formulae.

$$\begin{aligned} (a \pm b)^2 &= (a \pm b)(a \pm b) \\ &= a^2 \pm ab \pm ba + b^2 \\ &= a^2 \pm 2ab + b^2 \end{aligned}$$

$$\begin{aligned} (a+b)(a-b) &= a^2 - ab + ba - b^2 \\ &= a^2 - b^2 \end{aligned}$$

$$\begin{aligned} (a \pm b)^3 &= (a \pm b)^2(a \pm b) \\ &= (a^2 \pm 2ab + b^2)(a \pm b) \\ &= a^3 \pm a^2b \pm 2a^2b + 2ab^2 + ab^2 \pm b^3 \\ &= a^3 \pm 3a^2b + 3ab^2 \pm b^3 \end{aligned}$$

$$\begin{aligned} (a+b+c)^2 &= ((a+b)+c)^2 \\ &= (a+b)^2 + 2(a+b)c + c^2 \\ &= a^2 + 2ab + b^2 + 2ca + 2bc + c^2 \\ &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \end{aligned}$$

These formulae are important for factorisation.

[1] Expand and simplify.

(i) $(x-2y)(x+5y)$

(ii) $(x-4y)^2$

(iii) $(2x-3y+5z)^2$

(iv) $(x+3y-5z)(x+7y-5z)$

(v) $(x+1)(x+2)(x+3)(x+4)$

Example 1

Factorise.

(i) $2x^2 + x - 6$

(ii) $6x^2 - 11x + 5$

(iii) $x^3 + 8$

The extra factorisation formula:

$$acx^2 + (ad + bc - x + bd = (ax + b)(cx + d)$$

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$$

For finding constants a, b, c and d in the first formula, you can write down as

$$\begin{array}{ccccc} a & \times & b & \longrightarrow & bc \\ c & & d & \longrightarrow & ad \\ \hline & & & & ad + bc \end{array}$$

(i) $2x^2 + x - 6$
$ac = 2$, then we suppose that $a = 2$ and $c = 1$.
$bd = -6$, then we suppose that $b = -3$ and $d = 2$
Then $ad + bd = 4 - 3 = 1$, which is just the coefficient of the term x.
We can visualise these as:

$$\begin{array}{ccccc} 2 & \times & -3 & \longrightarrow & -3 \\ 1 & & 2 & \longrightarrow & 4 \\ \hline & & & & 1 \end{array}$$

Hence

$$2x^2 + x - 6 = (2x - 3)(x + 2)$$

(ii) $6x^2 - 11x + 5 = (x - 1)(6x - 5)$

$$\begin{array}{ccccc} 1 & \times & -1 & \longrightarrow & -6 \\ 6 & & -5 & \longrightarrow & -5 \\ \hline & & & & -11 \end{array}$$

(iii) $x^3 + 8 = x^3 + 2^3 = (x + 2)(x^2 - 2x + 4)$

[2] Factorise comletely.

(i) $x^2 - 4x + 3$

(ii) $2x^2 - 9x + 4$

(iii) $x^3 + 8$

(iv) $x^4 - 13x^2 + 36$

(v) $(x^2 + 3y^2)^2 - 4x^2y^2$

(vi) $x^6 - 1$

Example 2

Factorise.

(i) $2x^2 - 5xy - 3y^2 + 3x + 5y - 2$

(ii) $6x^2 + 5xy + y^2 + 2x - y - 20$

First make subject to x.

(i) $2x^2 - 5xy - 3y^2 + 3x + 5y - 2 = 2x^2 + (-5y + 3)x - (3y^2 - 5y + 2)$
First step, factorise $(3y^2 - 5y + 2)$.

$$\begin{array}{ccccc} 3 & \diagdown\!\!\!\!\diagup & -2 & \longrightarrow & -2 \\ 1 & & -1 & \longrightarrow & -3 \\ \hline & & & & -5 \end{array}$$

$2x^2 - 5xy - 3y^2 + 3x + 5y - 2 = 2x^2 + (-5y + 3)x - (3y - 2)(y - 1))$
Next step, considering the terms of y as constants and factorise.

$$\begin{array}{ccccc} 2 & \diagdown\!\!\!\!\diagup & y-1 & \longrightarrow & y-1 \\ 1 & & -(3y-2) & \longrightarrow & -6y+4 \\ \hline & & & & -5y+3 \end{array}$$

Then

$$\begin{aligned} 2x^2 - 5xy - 3y^2 + 3x + 5y - 2 &= (2x + (y - 1))(x - (3y - 2)) \\ &= (2x + y - 1)(x - 3y + 2) \end{aligned}$$

(ii)

$$6x^2 + 5xy + y^2 + 2x - y - 20 = 6x^2 + (5y+2)x + (y^2 - y - 20)$$

$$\begin{array}{ccccc} 1 & \times & -5 & \longrightarrow & -5 \\ 1 & & 4 & \longrightarrow & 4 \\ \hline & & & & -1 \end{array}$$

$$6x^2 + 5xy + y^2 + 2x - y - 20 = 6x^2 + (5y+2)x + (y-5)(y+4)$$

$$\begin{array}{ccccc} 3 & \times & y-5 & \longrightarrow & 2y-10 \\ 2 & & y+4 & \longrightarrow & 3y+12 \\ \hline & & & & 5y+2 \end{array}$$

Hence

$$\begin{aligned} 6x^2 + 5xy + y^2 + 2x - y - 20 &= 6x^2 + (5y+2)x + (y^2 - y - 20) \\ &= 6x^2 + (5y+2)x + (y-5)(y+4) \\ &= (3x + (y-5))(2x + (y+4)) \\ &= (3x + y - 5)(2x + y + 4) \end{aligned}$$

[3] Factorise comletely.

(i) $x^4 + x^2 + 1$

(ii) $4x^4 + 1$

(iii) $x^3 + 3xy + y^3 - 1$

Example 3

Express the following expression without surds.

$$3\sqrt{a^2} + 2\sqrt{a^2 + 4a + 4} - 2\sqrt{a^2 - 6a + 9}$$

where $0 < a < 3$.

Remind that

$$\sqrt{A^2} = |A| = \begin{cases} A & \text{when} A \geq 0 \\ -A & \text{when} A < 0 \end{cases}$$

$$3\sqrt{a^2} + 2\sqrt{a^2 + 4a + 4} - 2\sqrt{a^2 - 6a + 9}$$

$$= 3\sqrt{a^2 + 2\sqrt{(a+2)^2}} - 2\sqrt{(a-3)^2}$$

$$= 3|a| + 2|a+2| - 2|a-3|$$

$$= 3a + 2(a+2) - 2(-a+3)$$

$$= 7a - 2$$

[4] Express the following expressions without the modular sign.

(i) $|4|$
(ii) $|-5|$
(iii) $|x+1|$
(iv) $|x^2 - 4|$
(v) $|x^2 + 3|$

Example 4

Rationalise the denominator.

(i) $\dfrac{1}{\sqrt{3}}$

(ii) $\dfrac{1}{\sqrt{3}+\sqrt{2}}$

(iii) $\dfrac{1}{\sqrt{3}+\sqrt{2}-1}$

(iv) $\dfrac{1}{\sqrt[3]{4}+\sqrt[3]{3}}$

Using the expansion formulae:

$$(a+b)(a-b) = a^2 - b^2$$

and

$$(a \pm b)(a^2 \mp ab + b^2) = a^3 \pm b3$$

which is used for cubic radicals, as example (iv).

(i)

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

(ii)

$$\begin{aligned} \frac{1}{\sqrt{3}+\sqrt{2}} &= \frac{1}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} \\ &= \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2} \\ &= \frac{\sqrt{3}-\sqrt{2}}{3-2} \\ &= \sqrt{3}-\sqrt{2} \end{aligned}$$

(iii)

$$\begin{aligned}
\frac{1}{\sqrt{3}+\sqrt{2}-1} &= \frac{1}{\sqrt{3}+(\sqrt{2}-1)} \\
&= \frac{1}{\sqrt{3}+(\sqrt{2}-1)} \times \frac{\sqrt{3}-(\sqrt{2}-1)}{\sqrt{3}-(\sqrt{2}-1)} \\
&= \frac{\sqrt{3}-\sqrt{2}+1}{(\sqrt{3})^2-(\sqrt{2}-1)^2} \\
&= \frac{\sqrt{3}-\sqrt{2}+1}{3-(2-2\sqrt{2}+1)} \\
&= \frac{\sqrt{3}-\sqrt{2}+1}{2\sqrt{2}} \\
&= \frac{\sqrt{3}-\sqrt{2}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{6}-2+\sqrt{2}}{2\times 2} \\
&= \frac{-2+\sqrt{6}+\sqrt{2}}{4}
\end{aligned}$$

(iv)

$$\begin{aligned}
\frac{1}{\sqrt[3]{4}+\sqrt[3]{3}} &= \frac{1}{\sqrt[3]{4}+\sqrt[3]{3}} \times \frac{(\sqrt[3]{4})^2-\sqrt[3]{4}\cdot\sqrt[3]{3}+(\sqrt[3]{3})^2}{(\sqrt[3]{4})^2-\sqrt[3]{4}\cdot\sqrt[3]{3}+(\sqrt[3]{3})^2} \\
&= \frac{\sqrt[3]{16}-\sqrt[3]{12}+\sqrt[3]{9}}{(\sqrt[3]{4})^3+(\sqrt[3]{3})^3} \\
&= \frac{\sqrt[3]{16}-\sqrt[3]{12}+\sqrt[3]{9}}{4+3} \\
&= \frac{\sqrt[3]{16}-\sqrt[3]{12}+\sqrt[3]{9}}{7}
\end{aligned}$$

[5] Rationalise the denominator.

(i) $\dfrac{\sqrt{2}}{\sqrt{3}}$

(ii) $\dfrac{\sqrt{3}}{1-\sqrt{3}}$

(iii) $\dfrac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{\sqrt{2}-\sqrt{3}-\sqrt{5}}$

Example 5

Simplify the double radicals.

(i) $\sqrt{7+2\sqrt{10}}$

(ii) $\sqrt{12-4\sqrt{5}}$

(iii) $\sqrt{2+\sqrt{3}}$

The formula for simplifying the double radicals is

$$\sqrt{(a+b)\pm 2\sqrt{ab}} = \sqrt{a} \pm \sqrt{b} \qquad (a > b)$$

You can show this formula, when you evaluate the square of each side.
Note that not every double radicals can be simplified.

(i) $\sqrt{7+2\sqrt{10}} = \sqrt{(5+2)+2\sqrt{5\times 2}} = \sqrt{5}+\sqrt{2}$

(ii) $\sqrt{12-4\sqrt{5}} = \sqrt{12-2\sqrt{20}} = \sqrt{(10+2)-2\sqrt{10\times 2}} = \sqrt{10}-\sqrt{2}$

(iii)

$$\begin{aligned}\sqrt{2+\sqrt{3}} &= \frac{4+2\sqrt{3}}{\sqrt{2}} \\ &= \frac{\sqrt{(3+1)+2\sqrt{3}\times 1}}{\sqrt{2}} \\ &= \frac{\sqrt{3}+\sqrt{1}}{\sqrt{2}} \\ &= \frac{\sqrt{3}+1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{6}+\sqrt{2}}{2}\end{aligned}$$

[5] Simplify the double radicals.

(i) $\sqrt{5-2\sqrt{6}}$

(ii) $\sqrt{4+\sqrt{15}}$

(iii) $\sqrt{9-3\sqrt{24}}$

Example 6

(i) Write down the value of $125^{\frac{1}{3}}$ and $125^{-\frac{2}{3}}$.

(ii) Express $\dfrac{2x^2 - \frac{3}{2}x}{\sqrt{x}}$ in the form $2x^p - x^q$, where p and q are rational numbers.

(i) $125^{\frac{1}{3}} = (5^3)^{\frac{1}{3}} = 5$

$125^{-\frac{2}{3}} = (5^3)^{-\frac{2}{3}} = 5^{-2} = \frac{1}{25}$

(ii) $\dfrac{2x^2 - \frac{3}{2}x}{\sqrt{x}} = \left(2x^2 - \dfrac{3}{2}x\right) \times x^{-\frac{1}{2}} = 2x^{\frac{3}{2}} - \dfrac{3}{2}x^{\frac{1}{2}}$

[5] (i) Find the value of $8^{-\frac{4}{3}}$

(ii) Simplify $\dfrac{15x^{\frac{4}{3}}}{3x}$

(iii) Express $\dfrac{(3 - 4\sqrt{x})^2}{\sqrt{x}}$ in the form $ax^p + bx^q + c$, where $a,\ b,\ c,\ p$ and q are constants.

Exercise

[1] Arrange $a,\ b,\ c,\ d$ in the order from the smallest to the largest.

$$a=\sqrt{2+\sqrt{3}},\quad b=\sqrt{15}-2,\quad c=\frac{1}{\sqrt{5}-\sqrt{3}},\quad d=\sqrt{3}$$

[2] (1) Find the reminder when $(x^3+x^2+x+1)^2$ is divided by x^3-1.

(2) Find the remainder when $x^{3n}+1$, where n is a positive integer, is divided by x^3-1.

(3) Find the remainder when x^{20} is divided by x^3-1.

[3] Let $a,\ b,\ c$be a real number, different each other.

(1) Factorise $a^3b-ab^3+b^3c-bc^3+c^3a-ca^3$.

(2) Simplify $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}$.

[4] Simplify.

$$\frac{3x-5}{1-\dfrac{1}{1-\dfrac{1}{x+1}}}-\frac{x(2x-3)}{1+\dfrac{1}{1-\dfrac{1}{x-1}}}$$

[5] Find the value of a and the value of b when $(x-1)^2$ is a factor of x^n+ax+b $(x-1)^2$, where $n\geq 2$ is an integer.

[6] Is x^2+x+1 a facor of $(x^{100}+1)^{100}+(x^2+1)^{100}+1$?

[7] Let $x,\ y,\ z$ be real numbers which satisfy $xyz=1$.Show the following inequalies.

(1) $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\leqq x^2+y^2+z^2$

(2) $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}<\dfrac{1}{2}(x^2+y^2+z^2)^2$