Complex numbers and complex plane

Example 1

Write down the following numbers as the form $a + bi$, where a and b are real numbers and $i^2 = -1$.

(1) $(3 + 2i) + (5 - i)$

(2) $i - (6 + 2i)$

(3) $\sqrt{-3} \times \sqrt{-6}$

(4) $\dfrac{\sqrt{3}}{\sqrt{-7}}$

[1] Evaluate the following numbers.

(1) $(4 - 2i) + (5 - 2i)$

(2) $(5 - 2i) - (2 - 4i)$

(3) $(3 + 2i)(2 - 3i)$

(4) $\dfrac{3 + 2i}{3 - 2i}$

(5) $\sqrt{-5} \times \sqrt{-20}$

(6) $\dfrac{\sqrt{6}}{\sqrt{-2}}$

Example 2

Let $\overline{z}$ be the conjugate of $z = a + bi$, i.e. $\overline{z} = a - bi$.
Let $\alpha,\ \beta$ be complex numbers . Then prove the followng equalities.

(1) $\overline{\alpha \pm \beta} = \overline{\alpha} \pm \overline{\beta}$

(2) $\overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}$

(3) $\overline{\left(\dfrac{\alpha}{\beta}\right)} = \dfrac{\overline{\alpha}}{\overline{\beta}}$

[2] Let $\alpha = \dfrac{z}{1+z^2}$, where z is a complex number and $z \neq \pm\, i$.
Prove that α is a real number if $|z| = 1$.

[3] Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_0,\ a_1,\ \cdots, a_n$ are real numbers.
Prove that the equation $f(x) = 0$ has a root $\overline{\alpha}$, if α is a root of $f(x) = 0$.

Example 3

Put the following complex numbers to the complex plane. (Argand diagrams)
And write down as a polar coordinates.

(1) $-1+\sqrt{3}i$

(2) $1-i$

(3) $3i$

[4] Put the following complex numbers to the complex plane. (Argand diagrams)
And write down as a polar coordinates.

(1) $2+2i$

(2) $-i$

(3) $-\sqrt{3}+i$

Example 4

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Prove the following equalities.

(1) $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

(2) $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$

(3) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ (De Moivre's theorem)

[5] Evaluate the following expressions and write down as the polar coordinates.

(1) $2(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}) \times 3(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4})$

(2) $(\cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12})^6$

(3) $(-1 + \sqrt{3}i)^5$

(4) $\dfrac{3 + 3i}{-1 - \sqrt{3}i}$

Example 5

Let

$$z^5 = 1 \cdots (*)$$

(1) Solve the equation (*) by using the polar coordinates.

(2) Solve the equation (*) algebraically.

(3) Evaluate $\cos 72°$

[6] Solve the following equations, using the polar coordinates.

(1) $z^3 = 1$ (2) $z^4 = -1$

(3) $z^3 = 4\sqrt{3} + 4i$

[7] Using the De Moivre's theorem, prove that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

[8] Prove the fullowing equalities.

(1) $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \dfrac{\cos\frac{n}{2}\theta \sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}$

(2) $\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \dfrac{\sin\frac{n}{2}\theta \sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}$

Execises

[1] [1] Let $\alpha = \cos\dfrac{2\pi}{7} + i\sin\dfrac{2\pi}{7}$, $f(z) = z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$, where i is the imaginary unit.

(1) Using α, find the every roots of the equation $f(z) = 0$.

(2) Let $g(z) = (z-1)(z^2-1)(z^4-1)$,
and $h(z) = (z^3-1)(z^5-1)(z^6-1)$
Evaluate $g(\alpha) + h(\alpha)$ and $g(\alpha)h(\alpha)$.
Then evaluate $g(\alpha)$ and $h(\alpha)$.

[2] Using the results of [1], calculate

$$\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\sin\frac{6\pi}{7}$$

[2] Let z be a complex number, satisfyng $z^2 + \dfrac{1}{z^2} = \sqrt{3}$ and $Re(z) \geq 0,\ Im(z) \geq 0$.

(1) Find the argument and the absolute value of z.

(2) Evaluate $z + \dfrac{1}{z}$.

(3) Evaluate the area
of the quadrilateral OACB, whose vertices are O(0), A(z), B$\left(\dfrac{1}{z}\right)$, C$\left(z + \dfrac{1}{z}\right)$ on the complex plane.

[3] Given the triangle △ABC, whose vertices are A(z_1), B(z_2), C(z_3) on the complex plane.
And we have the condition :

$$(3-4i)z_1 + 4iz_2 - 3z_3 = 0$$

(1) Express $z_3 - z_1$ with z_1 and z_2.

(2) Find the ratio of three sides AB : BC : CA of the triangle and evaluate ∠BAC.

[4] Let A, B be represented by z_1, and z_2 on the complex plane. Suppose z_1 and z_2 satisfying

$z_2 = iaz_1$ (i is the imaginary unit, a is a real number satisfying $a > 0$)

and $|z_1| + |z_2| + |z_1 - z_2| = 1$.

(1) Express $|z_1|$ with a.

(2) Find the area of $\triangle$OAB, using a.

(3) Let $m = \sqrt{a} + \dfrac{1}{\sqrt{a}}$. Express the area of $\triangle$OAB with m.

(4) Find the maximum of the area of $\triangle$OAB, when a changes the range of $a > 0$.

[5] Let $\alpha,\ \beta,\ \gamma$ be different complex number each other, satisfying :

$$2\alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\alpha\gamma = 0$$

(1) Find the value of $\dfrac{\gamma - \alpha}{\beta - \alpha}$

(2) What type of triangle is $\triangle$ABC, whose vertices are A, B, C.

(3) Let $\alpha,\ \beta,\ \gamma$ be three roots of the cubic equation

$$x^3 + kx + 20 = 0$$

where k is a real constant. Find $\alpha,\ \beta,\ \gamma$ and k.

[6] Let $\{z_n\}$ be a sequence of complex numbers, which satisfies $z_1 = 3$ and $z_{n+1} = (1+i)z_n + i$ $(n \geq 1)$.

(1) Find z_n.

(2) Prove that $z_{8m-7} = 2^{5m-2} - 1$ for every positive integer m.

(3) Let P_n be a point on the complex plane, defined by the complex number z_n. Find the area

of the triangle whose vertices are $P_n,\ P_{n+1},\ P_{n+2}$.