Differential Calculus（1） ー Derivatives

Definition of a derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Formulae of derivatives

$$(f \pm g)' = f' \pm g'$$

$$(f \cdot g)' = f'g + fg'$$

$$(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$$

$$(g \circ f(x))' = g'(f(x))f'(x)$$

Derivatives of elementary functions

$$(x^\alpha)' = \alpha x^{\alpha-1}$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \frac{1}{\cos^2 x}$$

$$(e^x)' = e^x$$

$$(a^x)' = a^x \log a$$

$$(\log x)' = \frac{1}{x}$$

Formulae of derivatives - parametric functions, Inverse functions

Putting $x = f(t),\ y = g(t)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Taylor series

Suppose that $f(x)$ is infinitely differentiable in a neighbourhood of a, then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Especially when $a = 0$, the series is also called a Maclaurin series.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Example 1

Find the derivative for following functions.

(1) $y = x^4(x^2+1)$ (2) $y = \dfrac{2x}{x-1}$

(3) $y = \sqrt[4]{3x-5}$ (4) $y = \dfrac{1}{\sqrt{x-1}}$

[1] Find the derivative for following functions.

(1) $y = x^2(1-x)$ (2) $y = (x^2+2x-1)(x-x^2)$

(3) $y = (x-1)(x-2)(x-3)$ (4) $y = \dfrac{x+1}{x^2}$

(5) $y = \dfrac{x^2+1}{x^2-1}$ (6) $y = \dfrac{3x^2-2x+1}{x^2+2}$

(7) $y = (x^2+1)^7$ (8) $y = \dfrac{x^2}{(x+1)^3}$

(9) $y = \sqrt[3]{\dfrac{x-1}{x}}$ (10) $y = \sqrt[5]{\dfrac{x^2}{2x+1}}$

(11) $y = \dfrac{\sqrt{x-1}+\sqrt{x+1}}{\sqrt{x+1}-\sqrt{x-1}}$ (12) $y = \sqrt{\dfrac{(x-1)(x+3)}{(x+1)^3}}$

Example 2

Find the derivative for following functions.

(1) $y = \sin^3 x$

(2) $y = \cos 3x$

(3) $y = x^3 \sin x$

(4) $y = \sin(\cos x)$

(5) $y = \log 2x$

(6) $y = (x^2+1)e^{-3x}$

[2] Find the derivative for following functions.

(1) $y = \cos x - x$

(2) $y = x \cos^2 x$

(3) $y = \sin(\frac{\pi}{6} - 2x)$

(4) $y = \tan^2 x^3$

(5) $y = \cos^4 x^3$

(6) $y = x^2 \sin^3 2x$

(7) $y = \cos^3 x \sin x^3$

(8) $y = e^{\frac{x}{2}}$

(9) $y = x^2 e^{-x}$

(10) $y = \log \sqrt{x+1}$

(11) $y = e^x \log x$

(12) $y = \frac{x \log x}{\sin x^2}$

Example 3

[1] Take a logarithm of each side, and find the derivative for y

(1) $y = x^x$ (2) $y = \sqrt[3]{\dfrac{x+1}{x+4}}$

[2] Find the second derivative for following functions..

(1) $y = \sqrt{x+1}$ (2) $y = \log x$

[3] Find the derivative for following functions.

(1) $y = \sqrt[x]{x}$ (2) $y = x^{x^2}$

(3) $y = \dfrac{(x+1)^3}{(3x-1)^2(2x+1)^4}$ (4) $y = x^\alpha \log x$

[4] Find $f''(0)$, where $f(x) = \dfrac{1}{2}e^{x^2}$.

Differential Calculus (2) — Applications

Example 4

Sketch the graph of following functions.

(1) $y = \dfrac{2x^2 - x + 1}{x - 1}$ (2) $y = \dfrac{1}{x^2 + 4}$

(3) $y = x\sqrt{1 - x}$ (4) $y = \sqrt[3]{x^2(x - 2)}$

[5] Sketch the graph of following functions.

(1) $y = \dfrac{x^2 + 3x + 6}{x + 2}$ (2) $y = \dfrac{1}{x} - \dfrac{1}{x + 2}$

(3) $y = \dfrac{x^2}{\sqrt{1 - x}}$ (4) $y = \sqrt[3]{x^2(x + 2)}$

[6] Check local maxima and local minima of $f(x) = \dfrac{x}{(x - 1)^2}$, and sketch its graph. If exist find inflection points,

[7] Given $y = (x - 1)^{\frac{1}{3}}(x + 1)^{\frac{2}{3}}$.

(1) Find local maxima and local minima, and check the concavity of the function.

(2) Sketch the graph of this function.

Example 5

Sketch the graph of following functions.

(1) $y = \dfrac{\cos x}{1 - \cos x} \quad (0 \leqq x \leqq 2\pi)$

(2) $y = \dfrac{1}{e^x + x^{-x}}$

(3) $y = \log \dfrac{x-1}{x}$

(4) $x = \cos^3 t,\ y = \sin^3 t \quad (0 \leqq t \leqq 2\pi)$

[8] Sketch the graph of following functions. If needed, use the result $\lim_{x\to\infty} \dfrac{x}{e^x} = 0$ without prove.

(1) $y = \cos x - \dfrac{1}{2}\sin 2x \quad (0 \leqq x \leqq 2\pi)$

(2) $y = 2\sin x + \sin 2x \quad (0 \leqq x \leqq 2\pi)$

(3) $y = xe^{-x}$

(4) $y = x \log x$

[9] Check the region of increase and decrease, and concavity of the function $f(x) = (\dfrac{1}{x})^{\log x} \quad (x > 0)$, and sketch the graph of this function.

Example 6

Find the value a and b,when $y = e^{\frac{x}{3}}$ and $y = a\sqrt{2x-2}+b$ have the same tangent line at $x = 3$.

[10] (1) Find the equation of tangent of $y = x + \sin x$ at $(0, 0)$.

(2) Find the equation of tangent of $\sqrt{x} + \sqrt{y} = 1$, which passes at the point $(1, -\frac{1}{2})$.

Example 7

(1) How many different roots does the equation $e^x + x - 1 = 0$ have?

(2) Show the inequality $x \geqq \log(x+1) \geqq \dfrac{x}{x+1}$ when $x > -1$.

[11] (1) How many different roots does the equation $x^5 + 1 = k(x+1)^5$ have?

(2) Show the inequality $e^x > 1 + x$ when $x > 0$.

Example 8

(1) Find the Maclaurin series of $f(x) = e^x$

(2) Find the Maclaurin series of $f(x) = \arctan x$

[12] Give the Taylor polynomials of degree 3 at $x = 0$ for the following functions.

(1) $f(x) = \sin x - \cos x$

(2) $f(x) = e^x - e^{-x}$

(3) $f(x) = \tan x$

(4) $f(x) = \ln \dfrac{1+x}{1-x}$

Exercise

[1] Find derivative for following functions.

(1) $y = e^{-ax} \sin bx$, where a and b are constants.

(2) $y = x^3\sqrt{1+x^2}$, where $x > 0$.

[2] Find the tangent of $y = x\sqrt{1-x^2}$ at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right)$.

[3] Given the function $f(x) = \dfrac{4x^2+3}{2x-1}$.

(1) Find the derivative for $f'(x)$.

(2) Find local maxima and local minima of $f(x)$

(3) Find equations of asymptote of $y = f(x)$.

(4) Sketch the graph of $y = f(x)$

[4] Given a line $l : y = x + 1$ and a point P on the curve $y = \log x$. Find the coordinate of P, when the distance between P and l is minimum. And evaluate this minimum distance.

[5] Let $g(x)$ be the inverse function of $f(x) = e^{x-c}$, where c is a constant.

(1) Find $g(x)$

(2) How many points do $y = f(x)$ and $y = g(x)$ intersect each other ?

[6] Given $C : y = x^2$ and the line $l_t : y = tx - 1$ $(t > 0)$ which do not intersect each other.

(1) Evaluate the range of t.

(2) Let P_t be the point P on C and let Q_t be the point Q on l_t, when the distance between P and Q is minimum. Present the coordinates of P_t as function of t.

(3) Let $f(t)$ be length of the segment of line $\mathrm{P}_t\mathrm{Q}_t$ cutting by C. Find the minimum of $f(t)$, when t moves the range calculated at (1).

[7] (1) Show the inequality $e^x > 1 + x$ $(x > 0)$.

(2) Show the inequality $e^x > 1 + x + \frac{x^2}{2}$ $(x > 0)$.

(3) Show the inequality $e^x > \sum_{k=0}^{n} \frac{x^k}{k!}$ $(x > 0)$ by induction, where n is a positive integer.

(4) Show that $\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$, where n is a positive integer.

[8] (1) Using the addition formula for the tangent, prove that

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$$

(2) Prove that $\pi = 3.14159...$