Equations and Inequalities

Example 1

Solve the following equations.

(1) $3x + 5 = 4$

(2) $x - 2 = 3x + 1$

(3) $\frac{3}{4}x + \frac{1}{3} = 1$

(1)

$$\begin{aligned} 3x + 5 &= 4 \\ 3x &= 4 - 5 \\ 3x &= -1 \\ x &= -\frac{1}{3} \end{aligned}$$

(2)

$$\begin{aligned} x - 2 &= 3x + 1 \\ x - 3x &= 1 + 2 \\ -2x &= 3 \\ x &= -\frac{3}{2} \end{aligned}$$

(3) $\frac{3}{4}x + \frac{1}{3} = 1$

Multiply by 12

$$\begin{aligned} 9x + 4 &= 12 \\ 9x &= 12 - 4 \\ 9x &= 8 \\ x &= \frac{8}{9} \end{aligned}$$

[1] Solve the following equations.

(1) $x + 1 = -3x + 2$

(2) $2(2x - 1) = 3 = 5$

(3) $\frac{3x - 1}{2} = \frac{x + 4}{3}$

Example 2

Solve the following equations.

(1) $x^2-6x+5=0$

(2) $ax^2+bx+c=0, \quad a\neq 0$

(3) $ax^2+2b'x+c=0, \quad a\neq 0$

(1) If you find a factorisation, factorize the equation and solve it. It is easier.
But if you cannot find its factorisation, use the formulae (2),(3) of the roots of the quadratic equation.
$x^2-6x+5=0$
$(x-2)(x-3)=0$
$x=2,\ 3$

(2) $ax^2+bx+c=0, \quad (a\neq 0)$

$x^2+\frac{b}{a}x+\frac{c}{a}=0, \qquad (x^2+2\frac{b}{2a}x+(\frac{b}{2a})^2)-(\frac{b}{2a})^2+\frac{c}{a}=0, \qquad (x+\frac{b}{2a})^2=(\frac{b}{2a})^2-\frac{c}{a}$

$(x+\frac{b}{2a})^2=\frac{b^2}{4a^2}-\frac{c}{a}, \qquad (x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}, \qquad x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$

$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}, \qquad \mathbf{x}=\frac{\mathbf{-b\pm\sqrt{b^2-4ac}}}{\mathbf{2a}}$

(3) $ax^2+2b'x+c=0, \quad (a\neq 0)$

(2) より, $x=\frac{-2b'\pm\sqrt{(2b')^2-4ac}}{2a}=\frac{-2b'\pm\sqrt{4(b'^2-ac)}}{2a}=\frac{-2b'\pm 2\sqrt{b'^2-ac}}{2a}=\frac{-b'\pm\sqrt{b'^2-ac}}{a}$

すなわち, $\mathbf{x}=\frac{\mathbf{-b'\pm\sqrt{b'^2-ac}}}{\mathbf{a}}$

(2) and (3) are the quadratic formulae. And you need use them freely.

[2] Solve the following equations.

(1) $x^2+2x-8=0$

(2) $x^2+3x-5=0$

(3) $3x^2-4x-8=0$

(4) $3x^2+5x+2=0$

(5) $2x^2+2x+\ =0$

(6) $x^2-4x+4=0$

(7) $-0.2x^2+0.9x-1/1=0$

(8) $\sqrt{2}x^2-2\sqrt{6}x+6\sqrt{2}-4=0$

[3] Solve the following equations.

(1) $x^4-3x^2+2=0$

(2) $x^2-(2a+b)x+a^2+ab=0$

(3) $(x+3)^2-16=0$

(4) $\frac{x+3}{x-2}=3x-7$

Example 3

How many different real roots do the following equations have?

(1) $3x^2 + 4x + 8 = 0$

(2) $x^2 - 5x + 4 = 0$

(3) $x^2 - 6x + 9 = 0$

Discriminant：

$$D = b^2 - 4ac \quad , \quad \frac{D}{4} = b'^2 - ac$$

They are the parts of the quadratic formulae.

The quadratic equation has $\begin{cases} D > 0 & 2differentrealroots \\ D = 0 & 1realroot(doubleroot) \\ D < 0 & 2differentimaginaryroots \end{cases}$

(1) $3x^2 + 4x + 8 = 0$

$\frac{D}{4} = 2^2 - 3 \cdot 8 = 4 - 24 = -20 < 0$

Then the number of different real roots of the equation is 0.

(2) $x^2 - 5x + 4 = 0$

$D = (-5)^2 = 4 \cdot 1 \cdot 4 = 25 - 16 = 9 > 0$

Then the number of different real roots of the equation is 2.

(3) $x^2 - 6x + 9 = 0$

$\frac{D}{4} = (-3)^2 - 1 \cdot 9 = 9 - 9 = 0$

Then the number of different real roots of the equation is 1.

[4] How many different real roots do the following equations have?

(1) $x^2 + 18x + 81 = 0$

(2) $x^2 + bx + 7 = 0$

(3) $ax^2 + 5x - 2 = 0$

Example 4

Let α and β be the two roots of the equation

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

Write down $\alpha + \beta$ and $\alpha\beta$ with using $a,\ b,\ c$

Vieta's formulae to quadratic equations .

$ax^2 + bx + c = 0 \quad (a \neq 0)$ has 2 roots :

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Then

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

and

$$\alpha \cdot \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

A quadratic equation, whose 2 roots are $\alpha,\ \beta$, is

$$x^2 - (\alpha + \beta)x + \alpha \cdot \beta = 0$$

[5] Let α and β be the two roots of the equation.

$$x^2 + 3x - 5 = 0$$

Evaluate the following numbers.

(1) $\alpha + \beta$

(2) $\alpha\beta$

(3) $\alpha^2 + \beta^2$

(4) $\alpha^3 + \beta^3$

(5) $(\alpha - \beta)^2$

(6) $(\alpha^3 + \alpha^2 + 1)(\beta^3 + \beta^2 + 1)$

Example 5

Let $\alpha,\ \beta,\ \gamma$ be the three roots of the equation :

$$ax^3+bx^2+cx+d=0, \quad (a\neq 0)$$

Evaluate the following expressions with $a,\ b,\ c,\ d$.

(1) $\alpha+\beta+\gamma$

(2) $\alpha\beta+\beta\gamma+\gamma\alpha$

(3) $\alpha\beta\gamma$

Vieta's formulae to cubic equation :
$ax^3+bx^2+cx+d=0, \quad (a\neq 0)$ has 3 roots $\alpha,\ \beta,\ \gamma$, then we can factorize as

$$ax^3+bx^2+cx+d=a(x-\alpha)(x-\beta)(x-\gamma)$$

Then

$$ax^3+bx^2+cx+d=ax^3-a(\alpha+\beta+\gamma)x^2+a(\alpha\beta+\beta\gamma+\gamma\alpha)x-a\alpha\beta\gamma$$

Comparing coefficients,

$$b=-a(\alpha+\beta+\gamma), \quad c=a(\alpha\beta+\beta\gamma+\gamma\alpha), \quad d=-a\alpha\beta\gamma$$

Hence

$$\begin{cases} \alpha+\beta+\gamma &= -\dfrac{b}{a} \\ \alpha\beta+\beta\gamma+\gamma\alpha &= \dfrac{c}{a} \\ \alpha\beta\gamma &= -\dfrac{d}{a} \end{cases}$$

[6] Let $\alpha,\ \beta,\ \gamma$ be the three roots of the equation.

$$x^3-2x^2-4=0$$

(1) $\alpha^2+\beta^2+\gamma^2$

(2) $\alpha^3+\beta^3+\gamma^3$

(3) $\alpha^4+\beta^4+\gamma^4$

(4) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

[7] Let $\alpha,\ \beta,\ \gamma$ be the three roots of the equation :

$$x^3+3x^2-x+1=0$$

Find the cubic equations, whose roots are the following numbers.

(1) $\alpha+1,\ \beta+1,\ \gamma+1$

(2) $\dfrac{1}{\alpha-1},\ \dfrac{1}{\beta-1},\ \dfrac{1}{\gamma-1}$

Example 6

Solve the followng equations in $\mathbb{C}$

(1) $x^3 - 1 = 0$

(2) $12x^3 + 8x^2 - x - 1 = 0$

(3) $x^5 - 1 = 0$

(1) $x^3 - 1 = 0, \qquad (x-1)(x^2+x+1) = 0, \qquad x = 1,\ \dfrac{-1 \pm \sqrt{3}i}{2}$

(2) $12x^3 + 8x^2 - x - 1 = 0, \qquad (3x-1)(2x+1)^2 = 0, \qquad x = \dfrac{1}{3},\ -\dfrac{1}{2}$

(3) $x^5 - 1 = 0, \qquad (x-1)(x^4+x^3+x^2+x+1) = 0$

We shall solve the equation $x^4+x^3+x^2+x+1 = 0$.

Suppose $x \neq 0$ and multiply by $\dfrac{1}{x^2}$, we have $x^2 + x + 1 + \dfrac{1}{x} + \dfrac{1}{x^2} = 0, \quad (x+\dfrac{1}{x})^2 + (x+\dfrac{1}{x}) - 1 = 0$

Let $t = x + \dfrac{1}{x}$, then $t^2 + t - 1 = 0, \qquad t = \dfrac{-1 \pm \sqrt{5}}{2}$

Then $x + \dfrac{1}{x} = \dfrac{-1 \pm \sqrt{5}}{2}, \qquad 2x^2 + (1 \pm \sqrt{5})x + 2 = 0$

$x = \dfrac{-1+\sqrt{5} \pm \sqrt{10+2\sqrt{5}}i}{4},\ \dfrac{-1-\sqrt{5} \pm \sqrt{10-2\sqrt{5}}i}{4}$

Hence the roots of our equation are $x = 1,\ \dfrac{-1+\sqrt{5} \pm \sqrt{10+2\sqrt{5}}i}{4},\ \dfrac{-1-\sqrt{5} \pm \sqrt{10-2\sqrt{5}}i}{4}$

The roots of this equation are presented simply, if you use a polar system of complex numbers.

[8] Solve the followng equations in $\mathbb{C}$

(1) $x^3 - 7x + 6 = 0$

(2) $2x^3 - 5x^2 + 8x - 3 = 0$

(3) $x(x+1)(x=2) = 4 \cdot 5 \cdot 6$

(4) $x^4 - 8x^3 + 17x^2 - 8x + 1 = 0$

[9] If the equation $x^3 - 3x^2 - 12x + 3ax + 16 = 0$ has a positive real root a, find a and another roots.

[10] If the equation $x^3 + ax^2 + 9x + b = 0$ has a root $1 - 2i$, find the real coefficients $a,\ b$ and another roots of this equation.

[11] Let ω be one of the imaginary roots of $x^3 = 1$. Evaluate $\omega^2 + \omega$ and $\omega^{30} + \omega^{29} + \omega^{28}$.

Example 7

Solve the following inequalities.

(1) $5x+3<2$

(2) $6x^2-x-1<0$

(3) $x^2+6x-7>0$

(4) $x^2+x+1\leq 0$

(5) $x^2-6x+9\leq 0$

(1) $5x+3<2, \quad 5x<-1, \quad x<-\frac{1}{5}$

(2) $6x^2-x-1<0, \quad (3x+1)(2x-1)<0, \quad -\frac{1}{3}<x<\frac{1}{2}$

(3) $x^2+6x-7>0, \quad (x+7)(x-1)>0, \quad x<-7, 1<x$

(4) $x^2+x+1\leq 0, \quad (x+\frac{1}{2})^2+\frac{3}{4}\leqq 0$

As $(x+\frac{1}{2})^2+\frac{3}{4}\geqq\frac{3}{4}>0$, there don't exit such real numbers x.

(5) $x^2-6x+9\leq 0, \quad (x-3)^3\leqq 0$

As $(x-3)^3\geqq 0$, the solution of the inequality is $x=3$

[12] Solve the following inequalities.

(1) $6x-4>0$

(2) $-3x+9<0$

(3) $-x^2-4x+21>0$

(4) $2x^2+5x-8>0$

(5) $x^2-4x+4>0$

(6) $-x^2+2x-5>0$

Example 8

Solve the following inequalities.

(1) $x^3 - 3x + 2 > 0$

(2) $\dfrac{1}{x-2} \geq 1$

(3) $3|x-3| - 2|x+1| \geq 0$

(1) $x^3 - 3x + 2 > 0$
$(x-1)^2(x+2) > 0$
$x \neq 1$ and $x + 2 > 0$
Then $-2 < x < 1,\ 1 < x$

(2) $\dfrac{1}{x-2} \geq 1$
When you multiply by $(denominator)^2$, the direction of inequality will not be changed.
As $(denominator) \neq 0, x \neq 2$.
Multiply by $(x-2)^2$,
$\dfrac{(x-2)^2}{x-2} \geqq (x-2)^2$ $x - 2 \geqq (x-2)^2$ $x - 2 \geqq x^2 - 4x + 4$ $x^2 - 5x + 6 \leqq 0$ $(x-2)(x-3) \leqq 0$
Then $2 < x \leqq 3$

(3) $3|x-3| - 2|x+1| \geq 0$
i) When $x < -1$
$3(x-3) - 2(x+1) \geqq 0, \quad x - 11 \geqq 0, \quad x \geqq 11$
As $x < -1$, there do not exit the solution in this range.
ii) When $-1 \leqq x < 3$,
$3(x-3) - 2(-x-1) \geqq 0, \quad 5x - 7 \geqq 0, \quad x \geqq \dfrac{7}{5}$
As $-1 \leqq x < 3$, we have $\dfrac{7}{5} \leqq x < 3$
iii) When $3 \leqq x$,
$3(-x+3) - 2(-x-1) \geqq 0, \quad -x + 11 \geqq 0, \quad x \leqq 11$
As $3 \leqq x$, we have $3 \leqq x \leqq 11$
Hence the solution of the inequality is $\dfrac{7}{5} \leqq x \leqq 11$

[13] Solve the following inequalities.

(1) $x^4 + 3x^3 - x - 3 \leq 0$

(2) $\dfrac{3}{x+1} > 5$

(3) $\dfrac{1}{x-2} \leq \dfrac{2}{x+3}$

(4) $x^2 + 3|x| - 4 \leq 0$

(5) $|x^2 - 9| < x + 1$

[14] Solve the following systems of inequalities.

(1) $\begin{cases} x^2 + 2x - 3 & < & 0 \\ 2x^2 - 7x - 4 & \geq & 0 \end{cases}$

(2) $\begin{cases} x^2 - 2x + 3 & > & 0 \\ -x^2 + x + 4 & > & 0 \end{cases}$