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[1]

Assume that p is a negative real number. Given that three points $A(-1, 2, 0)$, $B(2, -2, 1)$ and $P(p, -1, 2)$ and let α be the plane passing through three points A, B and P.
Let Q be a point on the plane α such that $PQ \perp \alpha$.

(1) Express the coordinates of the point Q with p.

(2) Find the range of the value p such that the point Q is either in the triangle OAB or on the sides of the triangle OAB.

[2]

Given that $a_n = n(n+1)$, where n is a positive integer. Let d_n be the largest common divisor of a_n and a_{n+2}.

(1) Show that d_n is an even integer.

(2) Show that d_n is not divisible by 8.

(3) Given that a prime number p which is larger than or equal to 5. Show that d_n is not divisible by p.

(4) Show that $d_n \leq 12$ and find one example of n such that $d_n = 12$.

[3]

Let t be a real number such that $0 < t < 1$. Given that a function $f(x)$ defined at $x \neq 0,\ x \neq \dfrac{1}{t}$ as

$$f(x) = \frac{x+t}{x(1-tx)}$$

(1) Show that $f'(x)$ has exactly one local maximum and one local minimum.

(2) Let α and β be real numbers such that $f(\alpha)$ is the local maximum and $f(\beta)$ is the local minimum. Given that two points $P(\alpha, f(\alpha))$ and $Q(\beta, f(\beta))$. Find the equation of the loci of midpoint M of PQ, when t varies between $0 < t < 1$.

[4]

Let n be a integer larger than or equal to 3. There are two boxes X and Y and each box has n cards numbers from 1 to n.
The person A pick one card out from the box X, and the number on this card is the person A's point. The person B pick one card out from the box Y, and if the number on this card is between 3 and n then this number is the person B's point but if the number of the card is either 1 or 2, the card pull back into the box Y and pick again one card, whose number is the person B's point.

(1) Let $m \leq n$. Find the probability that the person B's point is m.

(2) Find the probability that the person B's point is larger than the person A's point.

[5]

Let $f(x)$ be a function which is continuous in the interval $0 \leq x \leq \pi$. And we define a sequence $f_1(x),\ f_2(x),\ \cdots$ as

$$f_1(x) = f(x)$$
$$f_{n+1}(x) = 2cosx + \frac{2}{\pi}\int_0^{\pi} f_n(t)\sin(x-t)\,dt \quad (n = 1,\ 2,\ 3,\ \cdots)$$

And given that

$$a_n = \frac{2}{\pi}\int_0^{\pi} f_n(t)\sin t\,dt, \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f_n(t)\cos t\,dt$$

(1) Express a_{n+1} and b_{n+1} with a_n and b_n.

(2) Let $c_n = a_n - 1$. Show that $c_{n+2} = -c_n$ and therefore express c_n with a_1 and b_1.

(3) Find one example of $f(x)$ such that a_n and b_n are independent to the value n.