5 Hyperbolic Functions

5.1 Hyperbolic Functions

The hyperbolic functions are defined as:

Definition

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

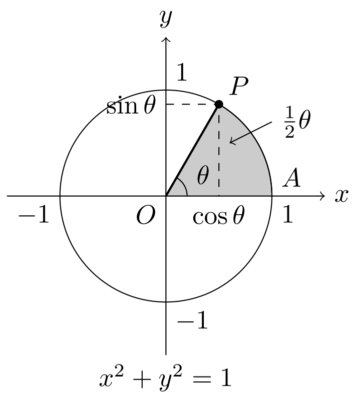
cosh means "hyperbolic cosine", **sinh** means "hyperbolic sine" and **tanh** means "hyperbolic tangent".

We may also define as:

$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{cosech} x = \frac{1}{\sinh x}, \quad \coth x = \frac{1}{\tanh x}$$

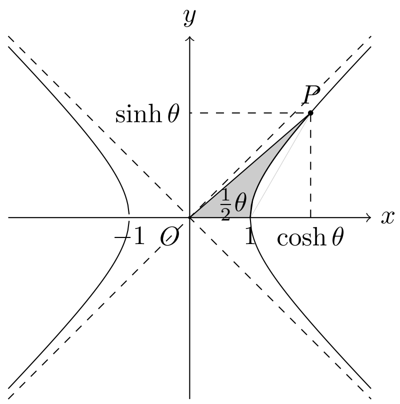
As their names, the hyperbolic functions resemble the trigonometric functions.

The trigonometric functions are defined on the unit circle $x^2 + y^2 = 1$ as:



In the diagram above, let A be the point whose coordinates are $(1, 0)$, and let P be a point on the circle $x^2 + y^2 = 1$ such that $\angle POA = \theta$.
(or we may say that let P be a point on the circle $x^2 + y^2 = 1$ such that the area of the sector OPA is $\frac{1}{2}\theta$)
Then we define $\cos\theta$ as the x-coordinate of the point P
and $\sin\theta$ as the y-coordinate of the point P.

The hyperbolic functions are defined on the hyperbolic function $x^2 - y^2 = 1$ as:



$x^2 - y^2 = 1$

In the diagram above, let P be a point on the hyperbolic curve $x^2 - y^2 = 1$ such that the area of the region surrounded by the segment OP, the x-axis and the curve $x^2 = y^2 = 1$ is $\dfrac{1}{2}\theta$.

Then we define $\cosh\theta$ as the x-coordinate of the point P and $\sinh\theta$ as the y-coordinate of the point P.

Since the trigonometric functions are based on the circle $x^2 + y^2 = 1$,

$$\cos^2\theta + \sin^2\theta = 1$$

And the hyperbolic functions are based on the hyperbolic function $x^2 - y^2 = 1$, then

$$\cosh^2\theta - \sinh^2\theta = 1$$

Based on these formulae, the formulae about hyperbolic functions resemblethe formulae about trigonometric functions except their signs.

5.2 The graphical representations of Hyperbolic Functions

(I)

$$y = \cosh x = \frac{e^x + e^{-x}}{2}$$

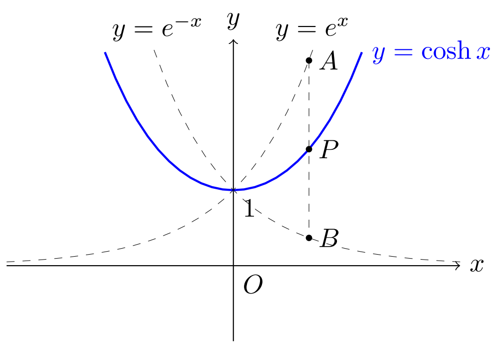
From the graphical representations of the curves $y = e^x$ and $y = e^{-x}$, we cam sketch the curve of $y = \cosh x$.

Let A and B be points on the curves $y = e^x$ and $y = e^{-x}$ respectively whose x-coordinate are the same. Let P be the midpoint of the segment AB, then the loci of the point P is the curve $y = \cosh x$.

Since

$$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \cosh x,$$

the function $\cosh x$ is an even function.
(Its graphical representation is symmetry with respect to the y-axis.)



(II)

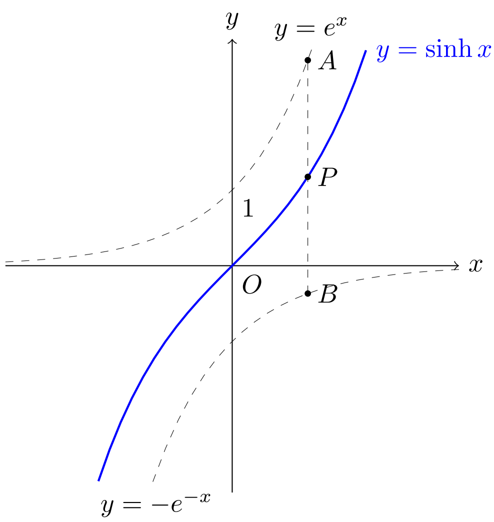
$$y = \sinh x = \frac{e^x - e^{-x}}{2}$$

Let A and B be points on the curves $y = e^x$ and $y = -e^{-x}$ respectively whose x-coordinate are the same. Let P be the midpoint of the segment AB, then the loci of the point P is the curve $y = \sinh x$.

Since

$$\sinh(-x) = \frac{e^{-x} - e^x}{2} = -\frac{e^x - e^{-x}}{2} = -\sinh x,$$

the function $\sinh x$ is an odd function.
(Its graphical representation is symmetry with respect to the origin O.)



(III)

$$y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Since

$$\begin{aligned}\lim_{x\to\infty} \tanh x &= \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ &= \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to-\infty} \tanh x &= \lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ &= \lim_{x\to-\infty} \frac{e^{2x} - 1}{e^{2x} + 1} \\ &= -1\end{aligned}$$

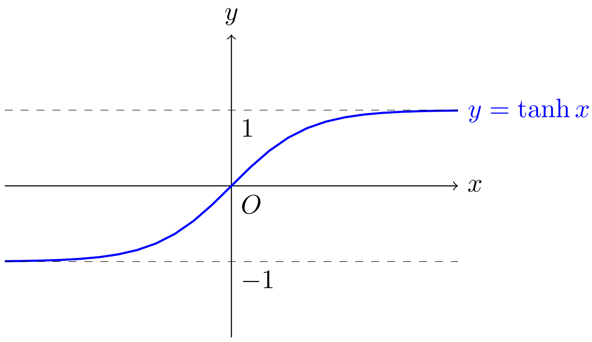
the lines $y = 1$ and $y = -1$ are asymptotes of the curve $y = \tanh x$.

$$\tanh(-x) = \frac{e^{-x} - e^x}{e^{-x} + e^x} = -\frac{e^x - e^{-x}}{e^x + e^{-x}} = -\tanh x$$

Then $\tanh x$ is an odd function.
(Its graphical representation is symmetry with respect to the origin O.)
As

$$\frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x} > 0,$$

$\tanh x$ is strictly increasing.



y
1
x
O
−1
y = tanh x

5.3 Formulae about Hyperbolic Functions

As the trigonometric functions,there are similar formulae for the hyperbolic functions.
The formulae 1 are the basic ones.

Formulae 1

$$\cosh^2 - \sinh^2 = 1,$$
$$1 - \tanh^2 x = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

Proof

$$\begin{aligned}
\cosh^2 - \sinh^2 &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4} \\
&= \frac{4}{4} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
1 - \tanh^2 x &= 1 - \frac{\sinh^2 x}{\cosh^2 x} \\
&= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\
&= \frac{1}{\cosh^2 x}
\end{aligned}$$

The formulae 2 look like the addition formulae and double angle formulae for trigonometric ones.

Formulae 2

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1$$

Proof

$$\begin{aligned}
&\sinh x \cosh y + \cosh x \sinh y \\
&\qquad = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} \\
&\qquad = \frac{e^x e^y + e^x e^{-y} - e^{-x} e^y - e^{-x} e^{-y}}{4} + \frac{e^x e^y - e^x e^{-y} + e^{-x} e^y - e^{-x} e^{-y}}{4} \\
&\qquad = \frac{e^x e^y - e^{-x} e^{-y}}{2} \\
&\qquad = \frac{e^{x+y} - e^{-(x+y)}}{2} \\
&\qquad = \sinh(x+y)
\end{aligned}$$

$$\begin{aligned}
&\cosh x \cosh y + \cosh x \cosh y \\
&\qquad = \frac{e^x + e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x - e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} \\
&\qquad = \frac{e^x e^y + e^x e^{-y} + e^{-x} e^y + e^{-x} e^{-y}}{4} + \frac{e^x e^y - e^x e^{-y} - e^{-x} e^y + e^{-x} e^{-y}}{4} \\
&\qquad = \frac{e^x e^y + e^{-x} e^{-y}}{2} \\
&\qquad = \frac{e^{x+y} + e^{-(x+y)}}{2} \\
&\qquad = \cosh(x+y)
\end{aligned}$$

$$\begin{aligned}
\sinh 2x &= \sinh(x+x) \\
&= \sinh x \cosh x + \cosh x \sinh x \\
&= 2 \sinh x \cosh x
\end{aligned}$$

$$\begin{aligned}
\cosh 2x &= \cosh(x+x) = \cosh^2 x + \sinh^2 x \\
&= (1 + \sinh^2 x) + \sinh^2 x = 1 + 2\sinh^2 x \\
&= 1 + 2(\cosh^2 x - 1) = 2\cosh^2 x - 1
\end{aligned}$$

From formulae about $\cosh 2x$, we can deduce the formulae:

Formulae 3

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

Looks like the half angle formulae for the trigonometric functions.

5.4 Inverse Hyperbolic Functions

Since hyperbolic functions are defined by the exponential functions, its inverse functions are defined by the inverse function of exponential functions: logarithmic functions.

Inverse hyperbolic functions

$$\begin{aligned}
\operatorname{arcosh} x &= \log(x+\sqrt{x^2-1}) \qquad (x \geq 1) \\
\operatorname{arsinh} x &= \log(x+\sqrt{x^2+1}) \\
\operatorname{artanh} x &= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) \qquad (|x| < 1)
\end{aligned}$$

Proof

Let $y = \operatorname{arcosh} x$, then $x = \cosh y$.

$$x = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + e^{-y}$$

$$e^{2y} - 2xe^y + 1 = 0$$

Then

$$e^y = x \pm \sqrt{x^2-1}$$

Since $y \geq 0$,

$$e^y = x + \sqrt{x^2-1}$$

Then

$$y = \ln(x+\sqrt{x^2-1})$$

Hence

$$\operatorname{arcosh} x = \ln(x+\sqrt{x^2-1}) \qquad (x \geq 1)$$

We leave another two inverse functions as an exercise:

Exercise:

Show that $\operatorname{arsinh} x = \ln(x+\sqrt{x^2+1})$ and $\operatorname{artanh} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

5.5 Differentiation and Integration of Hyperbolic Functions

Differentiation of hyperbolic functions

$$\begin{aligned}
\frac{d}{dx}(\cosh x) &= \sinh x \\
\frac{d}{dx}(\sinh x) &= \cosh x \\
\frac{d}{dx}(\tanh x) &= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x
\end{aligned}$$

Proof

$$\begin{aligned}
\frac{d}{dx}(\cosh x) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{e^x - e^{-x}}{2} \\
&= \sinh x
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \\
&= \frac{e^x + e^{-x}}{2} \\
&= \cosh x
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}(\tanh x) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) \\
&= \frac{(\sinh x)' \cosh x - \sinh x (\cosh x)'}{\cosh^2 x} \\
&= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\
&= \frac{1}{\cosh^2 x}
\end{aligned}$$

Differentiation of inverse hyperbolic functions

$$\begin{aligned}\frac{d}{dx}(\operatorname{arcosh} x) &= \frac{1}{\sqrt{x^2-1}}\\ \frac{d}{dx}(\operatorname{arsinh} x) &= \frac{1}{\sqrt{x^2+1}}\\ \frac{d}{dx}(\operatorname{artanh} x) &= \frac{1}{1-x^2}\end{aligned}$$

Proof

Let $y = \operatorname{arcosh} x$, then $x = \cosh y$.
Then

$$\begin{aligned}\frac{d}{dx}(\operatorname{arcosh} x) &= \frac{1}{\left(\frac{dx}{dy}\right)}\\ &= \frac{1}{\sinh y}\\ &= \frac{1}{\sqrt{\cosh^2 y - 1}}\\ &= \frac{1}{\sqrt{x^2-1}}\end{aligned}$$

Alternative proof:

$$\begin{aligned}\frac{d}{dx}(\operatorname{arcosh} x) &= \frac{d}{dx}(\log(x+\sqrt{x^2-1}))\\ &= \frac{1}{x+\sqrt{x^2-1}} \cdot \left(1 + \frac{2x}{2\sqrt{x^2-1}}\right)\\ &= \frac{1}{x+\sqrt{x^2-1}} \cdot \frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\\ &= \frac{1}{\sqrt{x^2-1}}\end{aligned}$$

Exercise:

Show that $\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{x^2+1}}$ and $\frac{d}{dx}(\operatorname{artanh} x) = \frac{1}{1-x^2}$.

Integration formulae 1

$$\begin{aligned}\int \cosh x\,dx &= \sinh x + C\\ \int \sinh x\,dx &= \cosh x + C\\ \int \tanh x\,dx &= \log(\cosh x) + C\end{aligned}$$

From the differentiation formulae we can deduce these integration formulae.

Integration formulae 2

$$\begin{aligned}\int \frac{1}{a^2+x^2}\,dx &= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \quad (|x|<a)\\ \int \frac{1}{a^2-x^2}\,dx &= \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right) + C = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| \quad (|x|<a)\\ \int \frac{1}{\sqrt{a^2-x^2}}\,dx &= \arcsin\left(\frac{x}{a}\right) + C\\ \int \frac{1}{\sqrt{x^2+a^2}}\,dx &= \operatorname{arsinh}\left(\frac{x}{a}\right) + C = \log(x+\sqrt{x^2+a^2}) + C\\ \int \frac{1}{\sqrt{x^2-a^2}}\,dx &= \operatorname{arcosh}\left(\frac{x}{a}\right) + C = \log(x+\sqrt{x^2-a^2}) + C \quad (x>a)\end{aligned}$$

When you differentiate the right hand side of each formula, you will get the function of left hand side.

When you integrate directly such functions, for example, if you find the term $x^2 + a^2$ in your function, you will try to use substitution $x = a\tan\theta$.

For example we shall calculate the integration $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx$.
Substitute $x = a\tan\theta$,

$$\frac{dx}{d\theta} = \frac{a}{\cos^2\theta}, \qquad dx = \frac{a d\theta}{\cos^2\theta}$$

Therefore

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2+a^2}}\,dx &= \int \frac{1}{\sqrt{a^2\tan^2\theta + a^2}}\,\frac{ad\theta}{\cos^2\theta} \\
&= \int \frac{1}{a\sqrt{\tan^2\theta + 1}}\,\frac{ad\theta}{\cos^2\theta} \\
&= \int \frac{1}{\sqrt{\frac{1}{\cos^2\theta}}}\,\frac{d\theta}{\cos^2\theta} \\
&= \int \frac{\cos\theta}{\cos^2\theta}\,d\theta \\
&= \int \frac{\cos\theta}{1-\sin^2\theta}\,d\theta
\end{aligned}
$$

Sunstitute $u = \sin\theta$,

$$
\frac{du}{d\theta} = \cos\theta, \qquad d\theta = \frac{du}{\cos\theta}
$$

Then

$$
\begin{aligned}
\int \frac{\cos\theta}{1-\sin^2\theta}\,d\theta &= \int \frac{\cos\theta}{1-u^2}\,\frac{du}{\cos\theta} \\
&= \int \frac{1}{1-u^2}\,du \\
&= \int \frac{1}{(1+u)(1-u)}\,du \\
&= \frac{1}{2}\int \left(\frac{1}{1+u} + \frac{1}{1-u}\right)\,du \\
&= \frac{1}{2}\log|1+u| - \log|1-u| + C \\
&= \frac{1}{2}\log\left|\frac{1+u}{1-u}\right| + C \\
&= \frac{1}{2}\log\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + C
\end{aligned}
$$

From $x = a\tan\theta$,

$$
\cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{1}{1+\frac{x^2}{a^2}} = \frac{a^2}{x^2+a^2}
$$

$$
\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{a^2}{x^2+a^2}} = \frac{x}{\sqrt{x^2+a^2}}
$$

Then

$$\begin{aligned}
\int \frac{\cos\theta}{1-\sin^2\theta}\, d\theta &= \frac{1}{2}\log\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + C \\
&= \frac{1}{2}\log\left|\frac{1+\frac{x}{\sqrt{x^2+a^2}}}{1-\frac{x}{\sqrt{x^2+a^2}}}\right| + C \\
&= \frac{1}{2}\log\left|\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}\right| + C \\
&= \frac{1}{2}\log\left|\frac{(\sqrt{x^2+a^2}+x)^2}{a^2}\right| + C \\
&= \log\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| + C \\
&= \log(\sqrt{x^2+a^2}+x) - \log a + C \\
&= \log(x+\sqrt{x^2+a^2}) + C
\end{aligned}$$

as C is an arbitrary constant.

If you find the term a^2-x^2 in your function, you will try substitution $x = a\sin\theta$.

For calculate the integral $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\, dx$,

Sunstitute $x = a\sin\theta$,

$$\frac{dx}{d\theta} = a\cos\theta, \qquad dx = a\cos\theta\ d\theta$$

Then

$$\begin{aligned}
\int \frac{1}{\sqrt{a^2-x^2}}\, dx &= \int \frac{1}{\sqrt{a^2-a^2\sin^2\theta}}\, a\cos\theta\ d\theta \\
&= \int \frac{a\cos\theta}{a\sqrt{1-\sin^2\theta}}\, d\theta \\
&= \int \frac{a\cos\theta}{a\sqrt{\cos^2\theta}}\, d\theta \\
&= \int d\theta \\
&= \theta + C \\
&= \arcsin\left(\frac{x}{a}\right) + C
\end{aligned}$$

5.6 Hyperbolic Functions vs Trigonometric Functions

When we use complex variables, we shall find the similarities between hyperbolic functions and trigonometric functions.

From the Euler's formulae:

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

Then we have

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

and

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

From this result, we shall define the trigonometric functions of a complex variables as:

Definition

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \quad \text{and} \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

where z is a complex number.

These $\cos z$ and $\sin z$ satisfy the formulae about the real variable trigonometric functions, as well as the formulae about the differentiation and the integration.

Example 1

Show that

$$\cos(ix) = \cosh x \quad \text{and} \quad \sin(ix) = i \sinh x$$

Answer:

$$\cos(ix) = \frac{1}{2}(e^{i(ix)} + e^{-i(ix)}) = \frac{1}{2}(e^{-x} + e^{x}) = \cosh x$$

and

$$\sin(ix) = \frac{1}{2i}(e^{i(ix)} - e^{-i(ix)}) = \frac{1}{2i}(e^{-x} - e^{x}) = -\frac{1}{i}\sinh x = i \sinh x$$

We define the hyperbolic functions of a complex variables as:

Definition

$$\cosh z = \frac{1}{2}(e^z + e^{-z}) \quad \text{and} \quad \sinh z = \frac{1}{2}(e^z - e^{-z})$$

where z is a complex number.

Example 2

Show that

$$\cosh(ix) = \cos x \quad \text{and} \quad \sinh(ix) = i \sin x$$

Answer

$$\cosh(ix) = \frac{1}{2}(e^{ix} + e^{-ix}) = \cos x$$

and

$$\sinh(ix) = \frac{1}{2}(e^{ix} - e^{-ix}) = i \cdot \frac{1}{2i}(e^{ix} - e^{-ix}) = i \sin x$$

Example 3

Show that

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$$
$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

Answer

$$\begin{aligned} \cos(x + iy) &= \cos x \cos(iy) - \sin x \sin(iy) \\ &= \cos x \cosh y - \sin x (i \sinh y) \\ &= cos x \cosh y - i \sin x \sinh y \end{aligned}$$

$$\begin{aligned} \sin(x + iy) &= \sin x \cos(iy) + \cos x \sin(iy) \\ &= \sin x \cosh y + \cos x (i \sinh y) \\ &= \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

In this example 3, if you put $x = 0$, you will find the result of the example 1.

Exercise

[1] Show that the following formulae:

(i) $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

(ii) $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$

(iii) $\cosh 3x = -3\cosh x + 4\cosh^3 x$

(iv) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

[2] Show, by using substitution $x = a\sinh u$, that the integration formula

$$\int \frac{1}{\sqrt{x^2+a^2}} = \operatorname{arsinh}\left(\frac{x}{a}\right) + C$$

[3] Solve the equation

$$4\cosh x + \sinh x = 8$$

giving your answer as natural logarithms.

[4] (i) Starting from the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials, prove that

$$\cosh 2x = 1 + 2\sinh^2 x$$

(ii) Solve the equation

$$\cosh 2x - 3\sinh x = 15$$

giving your answers as exact logarithms.

[5] (i) Find the values of a, b and c such that

$$4x^2 + 4x + 17 = a(x+b)^2 + c$$

(ii) Find the exact value of

$$\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{4x^2+4x+17}\,dx$$

[6]

$$f(x) = 5\cosh x - 4\sinh x$$

(i) Show that $f(x) = \dfrac{1}{2}(e^x + 9e^{-x})$.

(ii) Solve the equation $f(x) = 5$

(iii) Show that

$$\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5\cosh x - 4\sin x}\, dx = \frac{\pi}{18}$$

[7] Given that $y = \sinh^{n-1} x \cosh x$.

(i) Show that $\dfrac{dy}{dx} = (n-1)\sinh^{n-2} x + n\sinh^n x$.

I_n is defined by

$$I_n = \int_0^{\text{arsinh } 1} \sinh^n x \, dx \quad \text{for} \quad n \geq 0$$

(ii) Using the result in part (i); or otherwise, show that

$$nI_n = \sqrt{2} - (n-1)I_{n-2}, \qquad (n \geq 2)$$

(iii) Hence find the value of I_4.

[8] In this question, you may use without proof the results

$$4\cosh^3 y - 3\cosh y = \cosh 3y \quad \text{and} \quad \text{arcosh}\, y = \ln(y + \sqrt{y^2 - 1})$$

Show that the equation $x^3 - 3a^2x = 2a^3\cosh T$ is satisfied by $2a\cosh\left(\dfrac{1}{3}T\right)$ and hence that, if $c^2 \geq b^3 > 0$, one of the roots of the equation $x^3 - 3bx = 2c$ is $u + \dfrac{b}{u}$, where $u = (c + \sqrt{c^2 - b^3})^{\frac{1}{3}}$.

Show that the other two roots of the equation $x^3 - 3bx = 2c$ are the roots of the quadratic equation $x^2 + (u + \dfrac{b}{u})x + u^2 + \dfrac{b^2}{u^2} - b = 0$, and find these roots in terms of u, b and ω, where $\omega = \dfrac{1}{2}(-1 + i\sqrt{3})$.

Solve completely the equation $x^3 - 6x = 6$.

[9] The denite integrals T, U, V and X are dened by

$$T = \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t}\, dt \qquad U = \int_{\ln 2}^{\ln 3} \frac{u}{2\sinh u}\, du$$

$$V = -\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln v}{1-v^2}\, dv \qquad X = \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln(\coth x)\, dx$$

Show, without evaluating any of them, that T, U, V and X are all equal.

[10] In this question, a is a positive constant.

(i) Express $\cosh a$ in terms of exponentials.
By using partial fractions, prove that

$$\int_0^1 \frac{1}{x^2 + 2x\cosh a + 1}\, dx = \frac{a}{2\sinh a}$$

(ii) Find, expressing your answers in terms of hyperbolic functions,

$$\int_1^\infty \frac{1}{x^2 + 2x\sinh a - 1}\, dx$$

and

$$\int_0^\infty \frac{1}{x^4 + 2x^2\cosh a + 1}\, dx$$

[11] Show, by finding R and γ, that $A\sinh x + B\cosh x$ can be written in the form $R\cosh(x+\gamma)$ if $B > A > 0$. Determine the corresponding forms in the other cases that arise, for $A > 0$, according to the value of B.
Two curves have equations $y = \operatorname{sech} x$ and $y = a\tanh x + b$, where $a > 0$.

(i) In the case $b > a$, show that if the curves intersect then the x-coordinates of the points of intersection can be written in the form

$$\pm \operatorname{arcosh}\left(\frac{1}{\sqrt{b^2 - a^2}}\right) - \operatorname{artanh}\frac{a}{b}$$

(ii) Find the corresponding result in the case $a > b > 0$.

(iii) Find necessary and sufficient conditions on a and b for the curves to intersect at two distinct points.

(iv) Find necessary and sufficient conditions on a and b for the curves to touch and, given that they touch, express the y-coordinate of the point of contact in terms of a.