Integration（1）

Fomulae of integrals of elementary functions

$$\int x^{\alpha} dx = \frac{1}{\alpha+1} x^{\alpha+1} + C \quad (\alpha \neq -1)$$

$$\int \frac{1}{x} dx = \log|x| + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x} dx = \tan x + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\log a} + C$$

Integration by substitution

Putting $t = g(x)$

$$\int f(g(x)) \cdot g'(x) dx = \int f(t) dt$$

Integration by parts

$$\int f'(x) g(x) dx = f(x)g(x) - \int f(x) g'(x) dx$$

Example 1

Find indefinite integrals for following functions.

(1) $\int (3^x + x^3)dx$

(2) $\int \frac{x^2 - 1}{\sqrt[3]{x}}dx$

(3) $\int (2\sin 2x - \cos x + e^x)dx$

(4) $\int \sin^2 x dx$

[1] Find indefinite integrals for following functions.

(1) $\int \sqrt{x}(x+1)^2 dx$

(2) $\int (x^{\frac{2}{3}} - x^{-\frac{2}{3}})^2 dx$

(3) $\int (7^x + x^7 - \frac{1}{x})dx$

(4) $\int \frac{3x^3 - x^2}{x^2}dx$

(5) $\int \frac{\sqrt[4]{x^3} - x^2}{\sqrt{x}}dx$

(6) $\int (\sin x - 3\cos x)dx$

(7) $\int (x^2 - \frac{1}{\cos^2 x})dx$

(8) $\int (5\cos x + e^x)dx$

(9) $\int \frac{2\cos^3 x - 1}{\cos^2 x}dx$

(10) $\int (\sin^2 x - 1)dx$

(11) $\int \cos x(3 + 2\tan x)dx$

(12) $\int \frac{\cos^2 x}{1 + \sin x}dx$

Example 2

Find indefinite integrals for following functions.

(1) $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt{x+1}}dx$

(2) $\displaystyle\int \frac{1}{4x^2-1}dx$

(3) $\displaystyle\int \sin x \cos 2x dx$

(4) $\displaystyle\int \frac{x-1}{\sqrt[3]{x}-1}dx$

[2] Find indefinite integrals for following functions.

(1) $\displaystyle\int \frac{1}{\sqrt{x+1}-\sqrt{x}}dx$

(2) $\displaystyle\int \frac{x}{\sqrt{x^2+1}-x}dx$

(3) $\displaystyle\int \frac{2x^2-3x-2}{2x^2+3x+1}dx$

(4) $\displaystyle\int \frac{2x^2+x-1}{x^3+1}dx$

(5) $\displaystyle\int \frac{x^4+x^2+x-2}{x^2+2}dx$

(6) $\displaystyle\int \frac{1}{x^2-4}dx$

(7) $\displaystyle\int \frac{1}{x^2-x-2}dx$

(8) $\displaystyle\int \frac{1}{x(x+1)(x+2)}dx$

(9) $\displaystyle\int \sin 3x \sin 2x dx$

(10) $\displaystyle\int \cos^4 x dx$

(11) $\displaystyle\int (\sin x+\cos x)^2 dx$

(12) $\displaystyle\int \tan^2 x dx$

Example 3

Find indefinite integrals for following functions.

(1) $\int (x^2+1)(x+2)^8 dx$

(2) $\int xe^{-x^2+1} dx$

(3) $\int \frac{1}{1+e^x} dx$

(4) $\int \sin^5 x \cos^2 x dx$

[3] Find indefinite integrals for following functions.

(1) $\int x\sqrt{x+1} dx$

(2) $\int x^2(x-1)^6 dx$

(3) $\int x^2 e^{x^3} dx$

(4) $\int \frac{e^{2x}}{e^x+1} dx$

(5) $\int \frac{x}{\sqrt{x^2-1}} dx$

(6) $\int \frac{1}{x \log x} dx$

(7) $\int \frac{(\log x)^3}{x} dx$

(8) $\int \frac{\sqrt[3]{\log x}}{x} dx$

(9) $\int x^2 \sin(1-x^3) dx$

(10) $\int \frac{\cos x}{1+\sin x} dx$

(11) $\int \frac{\sin x}{\sqrt{\cos x}} dx$

(12) $\int \frac{1}{1+\sin x} dx$

Example 4

Find indefinite integrals for following functions.

(1) $\int \log x dx$

(2) $\int xe^{x+1} dx$

(3) $\int x \sin x dx$

(4) $\int e^x \cos x dx$

[4] Find indefinite integrals for following functions.

(1) $\int x \log x dx$

(2) $\int \frac{\log(x+1)}{x^2} dx$

(3) $\int x^n \log x dx$

(4) $\int \frac{\log x}{(x+1)^2} dx$

(5) $\int x^2 e^x dx$

(6) $\int (x+1)e^x dx$

(7) $\int x \cos x dx$

(8) $\int e^x \sin x dx$

(9) $\int x^2 \sin x dx$

(10) $\int x^3 \cos 2x dx$

(11) $\int e^x \cos^2 x dx$

(12) $\int \sin^4 x dx$

Example 5

Find the following integrals

(1) $\displaystyle\int_1^2 \frac{x^3-2x+4}{x^2}dx$

(2) $\displaystyle\int_0^1 \frac{1}{x^2-4}dx$

(3) $\displaystyle\int_0^{\frac{\pi}{2}} \cos x\sqrt{\sin x}dx$

(4) $\displaystyle\int_0^1 xe^x dx$

[5] Find the following integrals.

(1) $\displaystyle\int_0^1 \sqrt{x}dx$

(2) $\displaystyle\int_0^2 \frac{x^2-3x+1}{x+1}dx$

(3) $\displaystyle\int_1^2 \frac{1}{\sqrt{x}+\sqrt{x+2}}dx$

(4) $\displaystyle\int_0^{\pi} \cos^2 x dx$

(5) $\displaystyle\int_0^3 x\sqrt{3-x}dx$

(6) $\displaystyle\int_0^1 xe^{-\frac{x^2}{2}}dx$

(7) $\displaystyle\int_{-1}^0 x\cos(1-x^2)dx$

(8) $\displaystyle\int_1^e \frac{\sqrt{\log x}}{x}dx$

(9) $\displaystyle\int_0^1 xe^{-x}dx$

(10) $\displaystyle\int_0^3 \log(1+x)dx$

(11) $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x dx$

(12) $\displaystyle\int_1^e \frac{\log x}{x^3}dx$

Example 6

Find the following integrals

(1) $\int_0^1 \sqrt{1-x^2}dx$

(2) $\int_0^1 \frac{1}{3+x^2}dx$

(3) $\int_{-3}^2 |x^2-4|dx$

(4) $\int_0^\pi |\sin x+\cos x|dx$

[6] Find the following integrals

(1) $\int_2^4 \frac{1}{\sqrt{16-x^2}}dx$

(2) $\int_0^1 \sqrt{2-x^2}dx$

(3) $\int_0^{\sqrt{3}} \frac{1}{x^2+1}dx$

(4) $\int_0^{2\sqrt{3}} \frac{x^2}{4+x^2}dx$

(5) $\int_{-1}^1 \sqrt{|x|}dx$

(6) $\int_{-\pi}^{\frac{\pi}{2}} |\sin x|dx$

(7) $\int_{-1}^2 e^{|x|}dx$

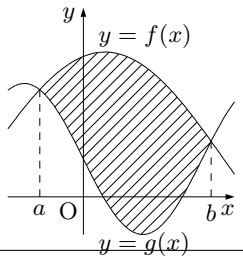
(8) $\int_0^{2\pi} |2\cos^2 x+\sin x-1|dx$

Integration（2） — Applications

Area

If $a \leqq x \leqq b$ and $f(x) \geqq g(x)$, the area S between two curves $y = f(x),\ y = g(x)$ and $x = a,\ x = b$ is

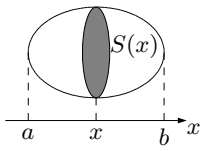
$$S = \int_a^b (f(x) - g(x))dx$$



Volume

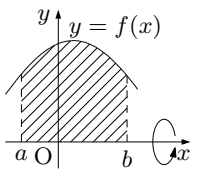
If the area of the intersection of a given solid by a plane $x = t$ is $S(x)$ $(a \leqq x \leqq b)$ then its volume V is

$$V = \int_a^b S(x)dx$$



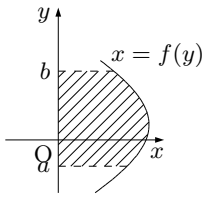
The volume V of revolution of the curve $y = f(x)$ $(a \leqq x \leqq b)$ around the x-axis is

$$V = \pi \int_a^b y^2 dx$$



The volume V of revolution of the curve $x = f(y)$ $(a \leqq y \leqq b)$ around the y-axis is

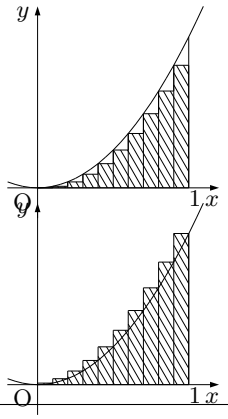
$$V = \pi \int_a^b x^2 dy$$



Relation between Integation and some limit of series

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\frac{k}{n}) = \int_0^1 f(x)dx$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n}) = \int_0^1 f(x)dx$$



Integration and inequality

Suppose that $a \leqq x \leqq b$.

If $f(x) \leqq g(x)$, then

$$\int_a^b f(x)dx \leqq \int_a^b g(x)dx$$

Example 1

Find the area enclosed by $\sqrt{x}+\sqrt{y}=\sqrt{a}$ $(a>0)$, x-axis and y-axis.

[1] Find the area enclosed by the following curves and x-axis.

(1) $y=\dfrac{x}{x+1},\ x=-2,\ x=2$

(2) $y=\sqrt{|x|},\ x=-1,\ x=1$

(3) $y=\dfrac{e^x+e^{-x}}{2},\ x=-2,\ x=2$

(4) $y=\sin x+\cos x,\ x=0,\ x=\dfrac{\pi}{4}$

Example 2

(1) Find the equation of tangent of $y = xe^{-x}$ at its inflection point.

(2) Find the area enclosed by the curve of (1), its tangent at inflection point and y-axis.

[2] Find the area enclosed by the following curves.

(1) $y = x^3,\ x = y^3$

(2) $y = \sin x,\ y = \cos 2x\ (0 < x < 2\pi)$

(3) $x = 5\cos t,\ y = 3\sin t$

(4) $x = t - \sin t,\ y = 1 - \cos t\ (0 \leqq t \leqq 2\pi)$

[3] (1) Find the constant a when $y = ax^2$ and $y = \log x + \dfrac{1}{2}$ are tangent each other.

(2) When (1) happens, find the area enclosed by two curves of (1).

Example 3

(1) Find the volume of revolution of $y = \log x$ $(1 \leqq x \leqq e)$ around the x-axis.

(2) Find the volume of revolution of the region enclosed by $y = x^3$ and $x = y^2$ around the y-axis.

[4] Find the volume of revolution of the region enclosed by the following functions around the x-axis.

(1) $y = \dfrac{1}{\sqrt{x}},\ y = 0,\ x = 1,\ x = e$

(2) $y = \sin x$ $(0 \leqq x \leqq \pi),\ y = 0$

(3) $y = x^3,\ x = y^3$ $(x \geqq 0)$

(4) $y = e^{-|x|},\ y = 1 - |x|,\ x = \pm 1$

Example 4

Find the volume of revolution of the region between $x = \cos^3 t,\ y = \sin^3 t\ (0 \leqq t \leqq \pi)$ and the x-axis around the x-axis.

[5] Find the volume of revolution of the region between $x = t - \sin t,\ y = 1 - \cos t\ (0 \leqq t \leqq 2\pi)$ and the x-axis around the x-axis.

[6] Given a function $f(x) = \sqrt{x}e^{-\frac{x}{2}}\ (x \geqq 0)$.

(1) Check the region of increase and decrease, the critical points and concavity of $y = f(x)$, then sketch its graph. You can use the result $\lim_{x\to\infty} f(x) = 0$ without proving.

(2) Find the volume $V(a)$ of revolution of the region enclosed by $y = f(x)$, the x-axis and two lines $x = a$, $x = a + 1\ (a \geqq 0)$ around the x-axis. Then find the value a, when $V(a)$ is the maximum.

Example 5

[1] Find the limit of series :

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}})$$

[2] Prove the inequality :

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}<\log x<1+\frac{1}{2}+\cdots+\frac{1}{n-1}$$

[7] Find the limit of following series.

(1) $\lim_{n\to\infty} n(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots+\frac{1}{(2n-1)^2})$

(2) $\lim_{n\to\infty} \frac{1}{n^2}(\sqrt{1+\frac{1}{n}}+2\sqrt{1+\frac{2}{n}}+3\sqrt{1+\frac{3}{n}}+\cdots+n\sqrt{1+\frac{n}{n}})$

(3) $\lim_{n\to\infty} (\frac{1}{n}+\frac{1}{n+a^2}+\frac{1}{n+2a^2}+\cdots+\frac{1}{n+(n-1)a^2})$

[8] (1) When $f(x)=\frac{1}{x^2}$, prove the following inequality.

$$f(2)+f(3)+\cdots+f(n)<\int_1^n f(x)dx<f(1)+f(2)+\cdots+f(n-1)$$

(2) Using the result of (1), prove the following inequality.

$$1-\frac{1}{n}+\frac{1}{n^2}<\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}<2-\frac{1}{n}$$

[9] Prove the following inequalities

(1) $1<\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}dx<\frac{\pi}{2}$

(2) $\frac{1}{2}<\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^3}}dx<\frac{\pi}{6}$