Kyoto University (150min)

[1] (40pt)

Answer the following questions.

(1) Let $0 < \theta < \frac{\pi}{2}$. Find the value θ such that $\cos\theta$ is not a rational number, but both $\cos 2\theta$ and $\cos 3\theta$ are rational numbers.
You may use the result that if p is a prime number then $\sqrt{p}$ is not rational.

(2) Find the values of

(i) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{x}{\cos^2 x}\, dx$

(ii) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{\cos x}$

(1) Ler $x = \cos\theta$, then $0 < x < 1$ for $0 < \theta < \frac{\pi}{2}$.

$$\cos 2\theta = 2\cos^2\theta - 1 = 2x^2 - 1$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta = 4x^3 - 3x$$

Assume that x is not rational and that both $2x^2 - 1$ and $4x^3 - 3x$ are rational.
Since $2x^2 - 1$ is rational, $x^2 = \dfrac{(2x^2-1)+1}{2}$ is also rational.
Then $4x^2 - 3$ is rational.
Since $4x^3 - 3x = x(4x^2 - 3)$ is rational and x is not rational, the rational number $4x^2 - 3$ must be 0.

$$4x^2 - 3 = 0$$

$$x^2 = \frac{3}{4}$$

$$x = \frac{\sqrt{3}}{2} \quad \text{which is not rational, as required.}$$

$$\cos\theta = \frac{\sqrt{3}}{2}$$

Hence

$$\theta = \frac{\pi}{6}$$

(2) (i)

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{x}{\cos^2 x}\,dx &= \int_0^{\frac{\pi}{4}} x(\tan x)'\,dx \\
&= \Big[x\tan x\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\,dx \\
&= \frac{\pi}{4}\tan\frac{\pi}{4} - \Big[-\log|\cos x|\Big]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{4} + \log(\cos\frac{\pi}{4}) - \log(\cos 0) \\
&= \frac{\pi}{4} + \log\frac{\sqrt{2}}{2}
\end{aligned}$$

(ii)

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} &= \int_0^{\frac{\pi}{4}} \frac{\cos x}{\cos^2 x}\,dx \\
&= \int_0^{\frac{\pi}{4}} \frac{\cos x}{1-\sin^2 x}\,dx
\end{aligned}$$

Substitute $u = \sin x$, then $\dfrac{du}{dx} = \cos x$, therefore $dx = \dfrac{du}{\cos x}$.

And $\begin{array}{c|ccc} x & 0 & \to & \frac{\pi}{4} \\ \hline u & 0 & \to & \frac{1}{\sqrt{2}} \end{array}$

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} &= \int_0^{\frac{\pi}{4}} \frac{\cos x}{1-\sin^2 x}\,dx \\
&= \int_0^{\frac{1}{\sqrt{2}}} \frac{\cos x}{1-u^2}\,\frac{du}{\cos x} \\
&= \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{1-u^2}\,du \\
&= \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{(1+u)(1-u)}\,du \\
&= \frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} \left(\frac{1}{1+u} + \frac{1}{1-u}\right)\,du \\
&= \frac{1}{2}\left(\Big[\log|1+u| - \log|1-u|\Big]_0^{\frac{1}{\sqrt{2}}}\right) \\
&= \frac{1}{2}\left(\log\left(1+\frac{1}{\sqrt{2}}\right) - \log\left(1-\frac{1}{\sqrt{2}}\right)\right) \\
&= \frac{1}{2}\log\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} \\
&= \frac{1}{2}\log\frac{\sqrt{2}+1}{\sqrt{2}-1} \\
&= \frac{1}{2}\log(\sqrt{2}+1)^2 \\
&= \log(\sqrt{2}+1)
\end{aligned}$$

[2] (30pt)

Given that $f(x) = x^3 + 2x^2 + 2$. Find all integers n such that both $|f(n)|$ and $|f(n+1)|$ are prime numbers.

You see that either n or $n+1$ is even.
If $x = 2k$ is an even number,

$$f(2k) = (2k)^3 + 2(2k)^2 + 2 = 8k^3 + 8k^2 + 2 = 2(4k^3 + 4k^2 + 1)$$

is also an even number.
Then $|f(2k)|$ is prime, if and only if $4k^3 + 4k^2 + 1 = 1$ or $4k^3 + 4k^2 + 1 = -1$

Assume that $4k^3 + 4k^2 + 1 = -1$,

$$4k^3 + 4k^2 = -2, \qquad 4k^2(k+1) = -2, \qquad 2k^2(k+1) = -1$$

Since $2k^2(k+1)$ is even and that -1 is odd, there are no such integer k.

Assume that $4k^3 + 4k^2 + 1 = 1$,

$$4k^3 + 4k^2 = 0, \qquad 4k^2(k+1) = 0$$

$$k = 0, \quad \text{or} \quad k = -1$$

Then

$$2k = 0, \quad \text{or} \quad 2k = -2$$

Since

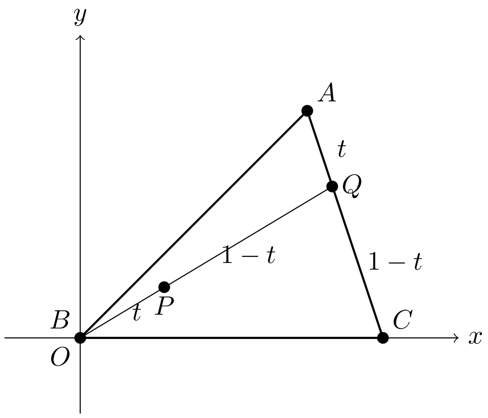
$$|f(-3)| = |(-3)^3 + 2(-3)^2 + 2| = |-27 + 18 + 2| = |-7| = 7, \quad \text{then prime.}$$
$$|f(-2)| = |(-2)^3 + 2(-2)^2 + 2| = |-8 + 8 + 2| = |2| = 2, \quad \text{then prime.}$$
$$|f(-1)| = |(-1)^3 + 2(-1)^2 + 2| = |-1 + 2 + 2| = |3| = 3, \quad \text{then prime.}$$
$$|f(0)| = |0^3 + 2 \cdot 0^2 + 2| = |0 + 0 + 2| = |2| = 2, \quad \text{then prime.}$$
$$|f(1)| = |1^3 + 2 \cdot 1^2 + 2| = |1 + 2 + 2| = |5| = 5, \quad \text{then prime.}$$

Hence the required integers n are

$$n = -3, \ -2, \ -1, \ 0$$

[3] (35pt)

Let S be the area of an acute triangle ABC. Let Q be a point which divides internally in the ratio $t : 1-t$ of the side AC and let P be a point which divides internally in the ratio $t : 1-t$ of the segment BQ, where t is a real number such that $0 < t < 1$. Find the area surrounded by the locus of the point P when t varies and the line BC with respect to S.



We fix the coordinates system such that $A(v, w)$, $B(0, 0)$ and $C(u, 0)$ where u, v and w are positive numbers.
Then

$$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ} = \overrightarrow{OA} + t\overrightarrow{AC} = \overrightarrow{OA} + t(\overrightarrow{OC} - \overrightarrow{OA}) = (1-t)\overrightarrow{OA} + \overrightarrow{OC}$$

Therefore

$$\overrightarrow{OQ} = (1-t)\begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} (1-t)v + u \\ (1-t)w \end{pmatrix}$$

And

$$\overrightarrow{OP} = t\overrightarrow{OQ}$$

Then

$$\overrightarrow{OP} = t\begin{pmatrix} (1-t)v + u \\ (1-t)w \end{pmatrix} = \begin{pmatrix} t(1-t)v + tu \\ t(1-t)w \end{pmatrix}$$

Therefore the equation of the locus of the point P is given as

$$\begin{cases} x = t(1-t)v + tu = (u+v)t - vt^2 \\ y = t(1-t)w = wt - wt^2 \end{cases}$$

$$\frac{dx}{dt} = (u+v) - 2vt, \qquad dx = ((u+v) - 2vt)\ dt$$

Hence the required area A is

$$\begin{aligned}
A &= \int_0^u y \, dx \\
&= \int_0^1 (wt - wt^2)((u+v) - 2vt) \, dt \\
&= \int_0^1 (w(u+v)t + (-w(u+v) - 2vw)t^2 + 2vwt^3) \, dt \\
&= \int_0^1 (w(u+v)t - w(u+3v)t^2 + 2vwt^3) \, dt \\
&= w\left[\frac{1}{2}(u+v)t^2 - \frac{1}{3}(u+3v)t^3 + \frac{1}{2}vt^4\right]_0^1 \\
&= w\left(\frac{1}{2}(u+v) - \frac{1}{3}(u+3v) + \frac{1}{2}v\right) \\
&= \frac{1}{6}uw
\end{aligned}$$

Since $S = \dfrac{1}{2}uw$,

$$A = \frac{1}{3}S$$

[4] (30pt)

When we throw a die n times and let $X_1,\ X_2,\ \cdots, X_n$ be the consecutive number of face of a die. Find the probability satisfying the following condition (I) with respect to n. We assume that $X_0 = 0$.

Condition (I): Given that $1 \leq k \leq n$. There exist one and only one k such that $X_{k-1} \leq 4$ and $X_k \geq 5$.

The condition (I) means:
Before k, every number of the face of die are less than or equal to 4, and after k , once X_{i+1} is less than or equal to 4, after that every number of the face of die must be less than or equal to 4.
Then

$$X_0 = 0,$$

$$X_1 \leq 4,\ X_2 \leq 4,\ \cdots,\ X_{k-1} \leq 4, \qquad \text{probability} = \left(\frac{4}{6}\right)^k$$

$$X_k \geq 5, x_{k+1} \geq 5,\ \cdots\ X_i \geq 5, \qquad \text{probability} = \left(\frac{2}{6}\right)^{i-k+1}$$

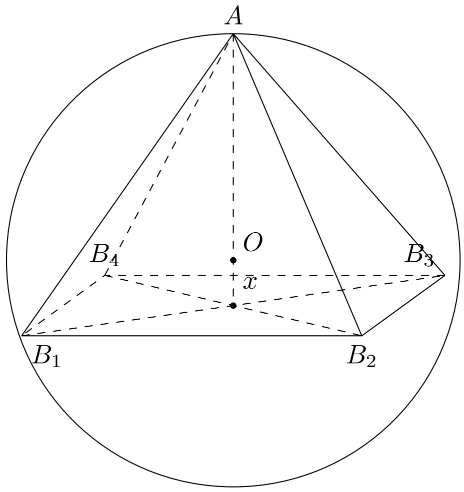
$$X_{i+1} \leq 4,\ X_{i+1} \leq 4,\ \cdots\ X_n \leq 4 \qquad \text{probability} = \left(\frac{4}{6}\right)^{n-i}$$

where $k \leq i \leq n$.
Therefore the required probability P is

$$\begin{aligned}
P &= \sum_{k=1}^{n}\sum_{i=k}^{n}\left(\frac{4}{6}\right)^{k-1}\left(\frac{2}{6}\right)^{i-k+1}\left(\frac{4}{6}\right)^{n-i} \\
&= \sum_{k=1}^{n}\sum_{i=k}^{n}\frac{2^{n+k-i-1}}{3^n} \\
&= \sum_{k=1}^{n}\left(\frac{2^{n+k-1}}{3^n}\sum_{i=k}^{n}2^{-i}\right) \\
&= \sum_{k=1}^{n}\left(\frac{2^{n+k-1}}{3^n}\frac{2^{-k}(1-2^{-(n-k+1)})}{1-2^{-1}}\right) \\
&= \sum_{k=1}^{n}\left(\frac{2^n(1-2^{-n+k-1})}{3^n}\right) \\
&= \sum_{k=1}^{n}\left(\frac{2^n-2^{k-1}}{3^n}\right) \\
&= n\left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n\frac{(1-2^n)}{1-2} \\
&= \frac{n\cdot 2^n - 2^n + 1}{3^n} \\
&= \frac{(n-1)2^n + 1}{3^n}
\end{aligned}$$

[5] (30pt)

Let five points A, B_1, B_2, B_3 and B_4 lie on the surface of the sphere whose radius is 1 and $B_1B_2B_3B_4$ forms a square which is the base of a pyramid $AB_1B_2B_3B_4$. Find the maximum volume of a pyramid $AB_1B_2B_3B_4$.



Let x be the distance from the center O of the sphere and the square $B_1B_2B_3B_4$.
Then the length of the diagonal of the square is given by $2\sqrt{1-x^2}$.
The area of the square $B_1B_2B_3B_4$ is

$$\frac{1}{2}(2\sqrt{1-x^2})^2 = 2(1-x^2)$$

The height of the pyramid $AB_1B_2B_3B_4$ is $1+x$.
Then the volume V of the pyramid $AB_1B_2B_3B_4$ is $1+x$ is

$$V = \frac{1}{3}\cdot 2(1-x^2)\cdot(1+x) = \frac{2}{3}(1+x-x^2-x^3)$$

$$\frac{dV}{dx} = \frac{2}{3}(1-2x-3x^2) = -\frac{2}{3}(3x-1)(x+1)$$

Then $\frac{dV}{dx} = 0$, when $x = -1,\ \frac{1}{3}$
Since $0 < x < 1$, the variation of V is

x	0		$\frac{1}{3}$		1
$\frac{dV}{dx}$		$+$	0	$-$	
V		$\nearrow$	$\frac{64}{81}$	$\searrow$	

Hence the maximum volume of a pyramid $AB_1B_2B_3B_4$ is $\frac{64}{81}$.

[6] (35pt)

Find the smallest positive integer n such that $(1+i)^n + (1-i)^n > 10^{10}$, where $i^2 = -1$.

Since

$$1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$$

and

$$1 - i = \sqrt{2}(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})$$

then

$$\begin{aligned}(1+i)^n + (1-i)^n &= \left(\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})\right)^n + \left(\sqrt{2}(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})\right)^n \\ &= 2^{\frac{n}{2}}(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}) + 2^{\frac{n}{2}}(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}) \\ &= 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}\end{aligned}$$

For $\cos\frac{n\pi}{4}$ is positive, $n = 8k,\ n = 8k+1$ or $n = 8k+7$, where k is a non-negative integer.

(i) When $n = 8k$,

$\cos\frac{n\pi}{4} = \cos\frac{8k\pi}{4} = \cos 2k\pi = 1$. Then

$$\begin{aligned}(1+i)^n + (1-i)^n &> 10^{10} \\ 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4} &> 10^{10} \\ 2^{\frac{8k}{2}+1} &> 10^{10} \\ 2^{4k+1} &> 10^{10} \\ \log_{10} 2^{4k+1} &> \log_{10} 10^{10} \\ (4k+1)\log_{10} 2 &> 10 \\ k &> \frac{1}{4}\left(\frac{10}{\log_{10} 2} - 1\right) \\ k &> 8.05\end{aligned}$$

Therefore the smallest integer k is $k = 9$.
Then $n = 8k = 72$.

(ii) When $n = 8k+1$,

$\cos\frac{n\pi}{4} = \cos\frac{(8k+1)\pi}{4} = \cos(2k\pi + \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.
Then

$$\begin{aligned}(1+i)^n + (1-i)^n &> 10^{10} \\ 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4} &> 10^{10} \\ 2^{\frac{8k+1}{2}+1}2^{-\frac{1}{2}} &> 10^{10} \\ 2^{4k+1} &> 10^{10}\end{aligned}$$

$$\log_{10} 2^{4k+1} > \log_{10} 10^{10}$$
$$(4k+1)\log_{10} 2 > 10$$
$$k > \frac{1}{4}\left(\frac{10}{\log_{10} 2} - 1\right)$$
$$k > 8.05$$

Therefore the smallest integer k is $k = 9$.
Then $n = 8k + 1 = 73$.

(iii) When $n = 8k + 7$,

$\cos\frac{n\pi}{4} = \cos\frac{(8k+7)\pi}{4} = \cos(2k\pi + \frac{7\pi}{4}) = \frac{1}{\sqrt{2}}$.
Then

$$(1+i)^n + (1-i)^n > 10^{10}$$
$$2^{\frac{n}{2}+1}\cos\frac{n\pi}{4} > 10^{10}$$
$$2^{\frac{8k+7}{2}+1} 2^{-\frac{1}{2}} > 10^{10}$$
$$2^{4k+4} > 10^{10}$$
$$\log_{10} 2^{4k+4} > \log_{10} 10^{10}$$
$$(4k+4)\log_{10} 2 > 10$$
$$k > \frac{1}{4}\left(\frac{10}{\log_{10} 2} - 4\right)$$
$$k > 7.30$$

Therefore the smallest integer k is $k = 8$.
Then $n = 8k + 7 = 71$.

Hence the smallest positive integer n is
$$n = 71$$