Entrance Exams of Kyoto University 2022

[1]

Show that

$$5.4 < \log_4 2022 < 5.5$$

You may use the result that $0.301 < \log_{10} 2 < 0.3011$.

Since $2000 < 2022 < 2048$,

$$\log_4 2000 < \log_4 2022 < \log_4 2048$$

$$\begin{aligned}
\log_4 2000 &= \frac{\log_{10} 2000}{\log_{10} 4} \\
&= \frac{\log_{10} 2 + \log_{10} 1000}{\log_{10} 2^2} \\
&= \frac{\log_{10} 2 + \log_{10} 10^3}{2\log_{10} 2} \\
&= \frac{\log_{10} 2}{2\log_{10} 2} + \frac{\log_{10} 10^3}{2\log_{10} 2} \\
&= \frac{1}{2} + \frac{3\log_{10} 10}{2\log_{10} 2} \\
&= \frac{1}{2} + \frac{3}{2\log_{10} 2}
\end{aligned}$$

Since $\log_{10} 2 < 0.3011$,

$$\frac{3}{2\log_{10} 2} > \frac{3}{2 \times 0.3011} > \frac{3}{0.6022}$$

Then

$$\frac{1}{2} + \frac{3}{2\log_{10} 2} > 0.5 + \frac{3}{0.6022} = 0.5 + 4.98\cdots = 5.48\cdots > 5.4$$

i.e. $\log_4 2000 > 5.4$.
Hence

$$5.4 < \log_4 2000 < \log_4 2022$$

And

$$\begin{aligned}\log_4 2048 &= \frac{\log_{10} 2048}{\log_{10} 4} \\ &= \frac{\log_{10} 2^{11}}{\log_{10} 2^2} \\ &= \frac{11 \log_{10} 2}{2 \log_{10} 2} \\ &= \frac{11}{2} = 5.5\end{aligned}$$

Then

$$\log_4 2022 < \log_4 2048 = 5.5$$

Therefore

$$5.4 < \log_4 2022 < 5.5$$

[2]

In a box there are n cards that each card is written one of the numbers from 1 to n. Suppose that $n \geq 5$ and each card has a different number. Pick three cards randomly from the box and let $X,\ Y$ and Z the numbers on these three cards, where $X < Y < Z$.
Find the probability such that $Y - X \geq 2$ and $Z - Y \geq 2$.

The complimentary event such that $Y - X \geq 2$ and $Z - Y \geq 2$ is the event such that

$$Y - X < 2 \quad \text{or} \quad Z - Y < 2$$

The events such that $Y - X < 2$ are

$$(1,2,3),\ (1,2,4),\ (1,2,5), \cdots, (1;2,n)$$
$$(2,3,4),\ (2,3,5), \cdots, (2,3,n)$$
$$\cdots\cdots\cdots$$
$$(n-2, n-1, n)$$

Then $(n-2)+(n-3)+\cdots+1 = \dfrac{1}{2}(n-1)(n-2)$ events.

And the events such that $Z - Y < 2$ are

$$(n, n-1, n-2),\ (n, n-1, n-3),\ (n, n-1, n-4), \cdots, (n, n-1, 1)$$
$$(n-1, n-2, n-3),\ (n-1, n-2, n-4), \cdots, (n-1, n-2, 1)$$
$$\cdots\cdots\cdots$$
$$(3,2,1)$$

Then $(n-2)+(n-3)+\cdots+1 = \dfrac{1}{2}(n-1)(n-2)$ events.

Among these events

$$(1,2,3),\ (2,3,4),\ \cdots, (n-2, n-1, n)$$

are counted twice. Then the number of events such that $Y - X < 2$ or $Z - Y < 2$ are

$$\frac{1}{2}(n-1)(n-2) + \frac{1}{2}(n-1)(n-2) - (n-2) = (n-1)(n-2) - (n-2) = (n-2)^2$$

Hence the probability such that $Y - X \geq 2$ and $Z - Y \geq 2$ is

$$1 - \frac{(n-2)^2}{{}^nC_3} = 1 - \frac{(n-2)^2}{\frac{n(n-1)(n-2)}{3!}} = 1 - \frac{6(n-2)^2}{n(n-1)(n-2)} = 1 - \frac{6(n-2)}{n(n-1)}$$
$$= \frac{n(n-1) - 6(n-2)}{n(n-1)} = \frac{n^2 - 7n + 12}{n(n-1)} = \frac{(n-3)(n-4)}{n(n-1)}$$

[3]

Let n be a natural number.
Find the greatest common divisor A_n of three integers $n^2+2,\ n^4+2$ and n^6+2.

Let d be a common factor of n^2+2 and n^4+2.
Then we can write that

$$n^2+2 = dX \quad \text{and} \quad n^4+2 = dY \quad \text{where } X \text{ and } Y \text{ are integers}$$

Since $n^4+2 = (n^2+2)(n^2-2)+6 = dY$,

$$dX(n^2-2)+6 = dY$$

$$d(Y - X(n^2-2)) = 6$$

Then d is a factor of 6, i.e. $d = 1,\ 2,\ 3$ or 6.
Therefore the GCD of $n^2+2,\ n^4+2$ and n^6+2 is $A_n = 1,\ 2,\ 3$ or 6.

i) When $n \equiv 0 \pmod{6}$,

$$n^2+2 \equiv 0^2+2 \equiv 2 \pmod{6}$$

$$n^4+2 \equiv 0^4+2 \equiv 2 \pmod{6}$$

$$n^6+2 \equiv 0^6+2 \equiv 2 \pmod{6}$$

Then we can write down as

$$n^2+2 = 6A+2 = 2(3A+1)$$

$$n^4+2 = 6B+2 = 2(3B+1)$$

$$n^6+2 = 6C+2 = 2(3C+1)$$

where $A,\ B$ and C are integers.
Since $3A+1,\ 3B+1$ and $3C+1$ are not divisible by 3,
the GCD of $n^2+2,\ n^4+2,\ n^66+2$ is 2.

ii) When $n \equiv \pm 1 \pmod{6}$,

$$n^2+2 \equiv 1^2+2 \equiv 3 \pmod{6}$$

$$n^4+2 \equiv 1^4+2 \equiv 3 \pmod{6}$$

$$n^6+2 \equiv 1^6+2 \equiv 3 \pmod{6}$$

Then we can write down as

$$n^2+2 = 6A'+3 = 3(2A'+1)$$

$$n^4 + 2 = 6B' + 3 = 3(2B' + 1)$$

$$n^6 + 2 = 6C' + 3 = 3(2C' + 1)$$

where A', B' and C' are integers.
Since $2A' + 1$, $2B' + 1$ and $2C' + 1$ are not divisible by 2, the GCD of $n^2 + 2$, $n^4 + 2$, $n^6 + 2$ is 3.

iii) When $n \equiv \pm 2 \pmod{6}$,

$$n^2 + 2 \equiv 2^2 + 2 \equiv 0 \pmod{6}$$

$$n^4 + 2 \equiv 2^4 + 2 \equiv 0 \pmod{6}$$

$$n^6 + 2 \equiv 2^6 + 2 \equiv 0 \pmod{6}$$

Then we can write down as

$$n^2 + 2 = 6A''$$

$$n^4 + 2 = 6B''$$

$$n^6 + 2 = 6C''$$

where A'', B'' and C'' are integers.
Then the GCD of $n^2 + 2$, $n^4 + 2$, $n^6 + 2$ is 6.

iv) When $n \equiv 3 \pmod{6}$,

$$n^2 + 2 \equiv 3^2 + 2 \equiv 5 \pmod{6}$$

$$n^4 + 2 \equiv 3^4 + 2 \equiv 5 \pmod{6}$$

$$n^6 + 2 \equiv 3^6 + 2 \equiv 5 \pmod{6}$$

Then we can write down as

$$n^2 + 2 = 6A''' + 5$$

$$n^4 + 2 = 6B''' + 5$$

$$n^6 + 2 = 6C''' + 5$$

where A''', B''' and C''' are integers.
Since $6A''' + 5$, $6B''' + 5$ and $6C''' + 5$ are not divisible by 2, 3 or 6, the GCD of $n^2 + 2$, $n^4 + 2$, $n^6 + 2$ is 1.

Hence

$$A_n = \begin{cases} 2 & (n = 6k) \\ 3 & (n = 6k \pm 1) \\ 6 & (n = 6k \pm 2) \\ 1 & (n = 6k + 3) \end{cases}$$

[4]

Given that a tetrahedron $OABC$, such that

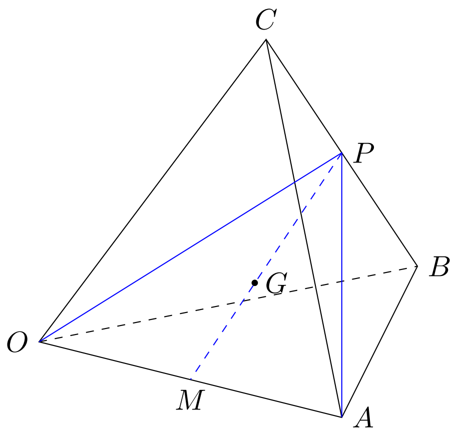
$$OA = 4,\ OB = AB = BC = 3 \text{ and } OC = AC = 2\sqrt{3}$$

Let P be a point on the side BC and let G be the centre of gravity of the triangle OAP.

(1) Show that $\overrightarrow{PG} \perp \overrightarrow{OA}$

(2) When the point P moves on the segment BC, find the minimum length of PG.

(1)



Let M be the midpoint of the segmentOA.

Then $\overrightarrow{PG} = \dfrac{2}{3}\overrightarrow{PM}$.

Since $\triangle OBC \equiv \triangle ABC$ (SSS condition),

$\angle OCB = \angle ACB$

Then $\triangle OCP \equiv \triangle ACP$ (SAS condition),

Therefore $PO = PA$

Hence the triangle POA is an isosceles triangle.

Since M is the midpoint of OA,

$OA \perp PM$

Hence $\overrightarrow{PG} \perp \overrightarrow{OA}$

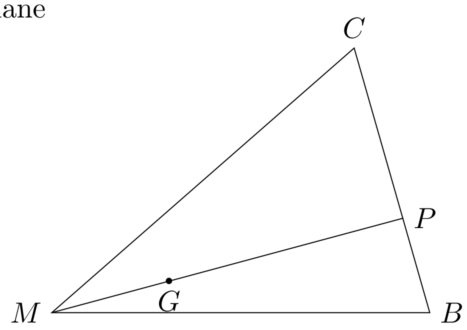
(Note: SSS (side-side-side) means that three corresponding sides of two triangles are equal in length.
SAS (side-angle-side) means that two corresponding sides of two triangles are equal in length and included angles are equal in measurement.
Both conditions are determinations of congruence of two triangles.)

(2) The points P and G are both on the same plane which consists the triangle BCM.
Then PG is minimum when $PG \perp BC$

$BM = \sqrt{AB^2 - AM^2} = \sqrt{3^2 - 2^2} = \sqrt{5}$

$CM = \sqrt{OC^2 - OM^2} = \sqrt{(2\sqrt{3})^2 - 2^2} = 2\sqrt{2}$

And $BC = 3$



Using the cosine rule we have

$$BC^2 = BM^2 + CM^2 - 2BM \cdot CM \cos\angle BMC$$

$$3^2 = (\sqrt{5})^2 + (2\sqrt{2})^2 - 2 \cdot \sqrt{5} \cdot 2\sqrt{2} \cos\angle BMC$$

$$\cos\angle BMC = \frac{(\sqrt{5})^2 + (2\sqrt{2})^2 - 3^2}{2 \cdot \sqrt{5} \cdot 2\sqrt{2}} = \frac{1}{\sqrt{10}}$$

Then

$$\sin\angle BMC = \sqrt{1 - \cos^2\angle BMC} = \sqrt{1 - \frac{1}{10}} = \frac{3}{\sqrt{10}}$$

Hence the area of the triangle BCM is

$$(\text{area of } \triangle BCM) = \frac{1}{2} BM \cdot CM \sin\angle BMC = \frac{1}{2} \cdot \sqrt{5} \cdot 2\sqrt{2} \cdot \frac{3}{\sqrt{10}} = 3$$

On the other side,

$$(\text{area of } \triangle BCM) = \frac{1}{2} BC \cdot PM = \frac{3}{2} PM$$

Then

$$\frac{3}{2} PM = 3$$

$$PM = 2$$

Hence the minimum length of PG is

$$PG = \frac{2}{3} PM = \frac{2}{3} \cdot 2 = \frac{4}{3}$$

[5]

Let S be the area of the region surrounded by the curve $C : y = \cos^3 x\ (0 \leq x \leq \frac{\pi}{2})$, x-axis and y-axis.
And let $f(t)$ be the area of the rectangle whose vertices are
$Q(t, \cos^3 t)\ (0 < t < \frac{\pi}{2})$, $O(0,0)$, $P(t,0)$ and $R(0, \cos^3 t)$.

(1) Find S.

(2) Prove that $f(t)$ has the maximum at only one value of t, say $t = \alpha$. and show that $f(\alpha) = \dfrac{\cos^4\alpha}{3\sin\alpha}$.

(3) Show that $\dfrac{f(\alpha)}{S} < \dfrac{9}{16}$.

(1)

$$\begin{aligned} S &= \int_0^{\frac{\pi}{2}} \cos^3 t dt \\ &= \int_0^{\frac{\pi}{2}} \cos^2 t \cos t dt \\ &= \int_0^{\frac{\pi}{2}} (1 - \sin^2 t) \cos t dt \end{aligned}$$

Substituting $u = \sin t$.
$\dfrac{du}{dt} = \cos t$, then $du = \cos t dt$, and

t	$0 \to \frac{\pi}{2}$
u	$0 \to 1$

$$\begin{aligned} S &= \int_0^1 (1 - u^2) du \\ &= \left[u - \frac{1}{3}u^3 \right]_0^1 \\ &= 1 - \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

(2) $f(t) = t\cos^3 t$, then

$$f'(t) = \cos^3 t - 3t\cos^2 t \sin t = \cos^2 t(\cos t - 3t\sin t)$$

Let $g(t) = \cos t - 3t\sin t$, then

$$g'(t) = -\sin t - 3\sin t - 3t\cos t = -(4\sin t + 3t\cos t)$$

Then for $0 < t < \frac{\pi}{2}$, $g'(t) < 0$.

It means that $g(t)$ is strictly decreasing in the interval $0 < t < \frac{\pi}{2}$.

$g(0) = 1 > 0$ and $g(\frac{\pi}{2}) = -\frac{3\pi}{2} < 0$

Then there is an unique value $t = \alpha$ such that $g(\alpha) = 0$.

The variation table of f is

t	0		α		$\frac{\pi}{2}$
$f'(t)$		$+$	0	$-$	
$f(t)$		$\nearrow$	(maximum)	$\searrow$	

Hence $f(t)$ has the maximum at only one value of t.

Since $g(\alpha) = 0$,

$$\cos\alpha - 3\alpha\sin\alpha = 0$$
$$\alpha = \frac{\cos\alpha}{3\sin\alpha}$$

Then

$$f(\alpha) = \alpha\cos^3\alpha = \frac{\cos\alpha}{3\sin\alpha}\cdot\cos^3\alpha = \frac{\cos^4\alpha}{3\sin\alpha}$$

(3) $\frac{f(\alpha)}{S} = \frac{\cos^4\alpha}{3\sin\alpha}\cdot\frac{3}{2} = \frac{\cos^4\alpha}{2\sin\alpha}$

Let $h(x) = \frac{\cos^4 x}{2\sin x}$,

$$h'(x) = \frac{-4\cos^3 x\sin^2 x - \cos^5 x}{2\sin^2 x} = -\frac{\cos^3 x(4\sin^2 x + \cos^2 x)}{2\sin^2 x}$$

In the interval $0 < x < \frac{\pi}{2}$, $h'(x) < 0$,

then $h(x)$ is strictly decreasing in the interval $0 < x < \frac{\pi}{2}$

Since $g(\frac{\pi}{6}) = \cos\frac{\pi}{6} - 3\cdot\frac{\pi}{6}\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{4} = \frac{2\sqrt{3}-\pi}{4} > 0$,

$\frac{\pi}{6} < \alpha$.

As $h(x)$ is strictly decreasing, $h(\frac{\pi}{6}) > h(\alpha)$

$$h(\frac{\pi}{6}) = \frac{\cos^4\frac{\pi}{6}}{2\sin\frac{\pi}{2}} = (\frac{\sqrt{3}}{2})^4 = \frac{9}{16}$$

Hence $h(\alpha) < \frac{9}{16}$, i.e.

$$\frac{f(\alpha)}{S} < \frac{9}{16}$$

[6]

Given that two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$x_1 = 0, \quad x_{n+1} = x_n + n + 2\cos(\frac{2\pi x_n}{3}) \quad (n = 1,\ 2,\ 3,\ \cdots)$$

$$y_{3m+1} = 3m, \quad y_{3m+2} = 3m + 2, \quad y_{3m+3} = 3m + 4 \quad (m = 0,\ 1,\ 2,\ )$$

Find the n-th term of the sequence $\{x_n - y_n\}$.

$$x_2 = x_1 + 1 + 2\cos(\frac{2\pi x_1}{3}) = 0 + 1 + 2\cos 0 = 3$$

$$x_3 = x_2 + 2 + 2\cos(\frac{2\pi x_2}{3}) = 3 + 2 + 2\cos 2\pi = 7$$

$$x_4 = x_3 + 3 + 2\cos(\frac{2\pi x_3}{3}) = 7 + 3 + 2\cos(\frac{14\pi}{3}) = 10 - 1 = 9$$

$$x_5 = x_4 + 4 + 2\cos(\frac{2\pi x_4}{3}) = 9 + 4 + 2\cos 6\pi = 13 + 2 = 15$$

$$x_6 = x_5 + 5 + 2\cos(\frac{2\pi x_5}{3}) = 15 + 5 + 2\cos 10\pi = 20 + 2 = 22$$

$$x_7 = x_6 + 6 + 2\cos(\frac{2\pi x_6}{3}) = 22 + 6 + 2\cos(\frac{44\pi}{3}) = 28 - 1 = 27$$

We can suppose that, for $m = 0,\ 1,\ 2,\ \cdots,$

$$x_{3m+1} \equiv 0 \quad (\text{mod. } 3)$$

$$x_{3m+2} \equiv 0 \quad (\text{mod. } 3)$$

$$x_{3m+3} \equiv 1 \quad (\text{mod. } 3)$$

Prove this suggestion by induction.
Since $x_1 = 0 \equiv 0, \quad x_2 = 3 \equiv 0$ and $x_3 = 7 \equiv 1$ for mod.3, then the suggestion is correct when $m = 0$.

Suppose that the suggestion is correct for m, then

$$x_{3(m+1)+1} = x_{3m+3} + (3m + 3) + 2\cos(\frac{2\pi x_{3m+3}}{3})$$

Since $x_{3m+3} \equiv 1$ (mod. 3), $\cos(\frac{2\pi x_{3m+3}}{3}) = -\frac{1}{2}$.
Then $x_{3m+3} + 2\cos(\frac{2\pi x_{3m+3}}{3}) = x_{3m+3} - 1 \equiv 0$ (mod. 3).
Therefore

$$x_{3(m+1)+1} = x_{3m+3} + (3m + 3) - 1 \equiv 0 \quad (\text{mod. } 3)$$

$$x_{3(m+1)+2} = x_{3m+4} + (3m+4) + 2\cos(\frac{2\pi x_{3m+4}}{3})$$

Since $x_{3m+4} = x_{3(m+1)+1} \equiv 0$ (mod. 3),

$$2\cos(\frac{2\pi x_{3m+4}}{3}) = 2 \cdot 1 = 2$$

Then $x_{3(m+1)+2} = x_{3m+4}+(3m+4)+2\cos(\frac{2\pi x_{3m+4}}{3}) \equiv 0+1+2 \equiv 3 \equiv 0 \pmod{3}$

$$x_{3(m+1)+3} = x_{3m+5} + (3m+5) + 2\cos(\frac{2\pi x_{3m+5}}{3})$$

Since $x_{3m+5} = x_{3(m+1)+2} \equiv 0$ (mod. 3),

$$2\cos(\frac{2\pi x_{3m+5}}{3}) = 2 \cdot 1 = 2$$

Then $x_{3(m+1)+3} = x_{3m+5}+(3m+5)+2\cos(\frac{2\pi x_{3m+5}}{3}) \equiv 0+2+2 \equiv 4 \equiv 1 \pmod{3}$

Therefore we showed that

$$x_{3(m+1)+1} \equiv 0 \pmod{3}$$

$$x_{3(m+1)+2} \equiv 0 \pmod{3}$$

$$x_{3(m+1)+3} \equiv 1 \pmod{3}$$

Hence we proved, by induction, that

$$x_{3m+1} \equiv 0 \pmod{3}$$

$$x_{3m+2} \equiv 0 \pmod{3}$$

$$x_{3m+3} \equiv 1 \pmod{3}$$

for $m = 0,\ 1,\ 2,\ \cdots$.

Now we shall see the n-th term of the sequence $\{x_n\}$.

$$
\begin{aligned}
x_{3m+2} &= x_{3m+1} + (3m+1) + 2\cos(\frac{2\pi x_{3m+1}}{3}) \\
&= \Big(x_{3m} + 3m + 2\cos(\frac{2\pi x_{3m}}{3})\Big) + (3m+1) + 2\cos(\frac{2\pi x_{3m+1}}{3}) \\
&= x_{3m} + 3m + (3m+1) + 2\Big(\cos(\frac{2\pi x_{3m}}{3}) + \cos(\frac{2\pi x_{3m+1}}{3})\Big) \\
&= \Big(x_{3m-1} + (3m-1) + 2\cos(\frac{2\pi x_{3m-1}}{3})\Big) \\
&\qquad + 3m + (3m+1) + 2\Big(\cos(\frac{2\pi x_{3m}}{3}) + \cos(\frac{2\pi x_{3m+1}}{3})\Big) \\
&= x_{3m-1} + (3m-1) + 3m + (3m+1) \\
&\qquad + 2\Big(\cos(\frac{2\pi x_{3m-1}}{3}) + \cos(\frac{2\pi x_{3m}}{3}) + \cos(\frac{2\pi x_{3m+1}}{3})\Big) \\
&= \cdots \\
&= x_1 + \sum_{k=1}^{3m+1} k + 2\sum_{k=1}^{3m+1} \cos(\frac{2\pi x_k}{3}) \\
&= 0 + \frac{(3m+1)(3m+2)}{2} + 2\Big((1+1-\frac{1}{2}) + (1+1-\frac{1}{2}) + \cdots + (1+1-\frac{1}{2}) + 1\Big) \\
&= \frac{(3m+1)(3m+2)}{2} + 2(\frac{3}{2}m + 1) \\
&= \frac{(3m+1)(3m+2)}{2} + 3m + 2
\end{aligned}
$$

$$
\begin{aligned}
x_{3m+3} &= x_{3m+2} + (3m+2) + 2\cos(\frac{2\pi x_{3m+2}}{3}) \\
&= \Big(x_{3m+1} + (3m+1) + 2\cos(\frac{2\pi x_{3m+1}}{3})\Big) + (3m+2) + 2\cos(\frac{2\pi x_{3m+2}}{3}) \\
&= x_{3m+1} + (3m+1) + (3m+2) + 2\Big(\cos(\frac{2\pi x_{3m+1}}{3}) + \cos(\frac{2\pi x_{3m+2}}{3})\Big) \\
&= \cdots \\
&= x_1 + \sum_{k=1}^{3m+2} k + 2\sum_{k=1}^{3m+2} \cos(\frac{2\pi x_k}{3}) \\
&= 0 + \frac{(3m+2)(3m+3)}{2} + 2\Big((1+1-\frac{1}{2}) + (1+1-\frac{1}{2}) + \cdots + (1+1-\frac{1}{2}) + 1 + 1\Big) \\
&= \frac{(3m+2)(3m+3)}{2} + 2(\frac{3}{2}m + 2) \\
&= \frac{(3m+2)(3m+3)}{2} + 3m + 4
\end{aligned}
$$

$$
\begin{aligned}
x_{3m+4} &= x_{3m+3} + (3m+3) + 2\cos(\frac{2\pi x_{3m+3}}{3}) \\
&= \Big(x_{3m+2} + (3m+2) + 2\cos(\frac{2\pi x_{3m+2}}{3})\Big) + (3m+3) + 2\cos(\frac{2\pi x_{3m+3}}{3}) \\
&= x_{3m+2} + (3m+2) + (3m+3) + 2\Big(\cos(\frac{2\pi x_{3m+2}}{3}) + \cos(\frac{2\pi x_{3m+3}}{3})\Big) \\
&= \cdots \\
&= x_1 + \sum_{k=1}^{3m+3} k + 2\sum_{k=1}^{3m+3} \cos(\frac{2\pi x_k}{3}) \\
&= 0 + \frac{(3m+3)(3m+4)}{2} + 2\Big((1+1-\frac{1}{2}) + (1+1-\frac{1}{2}) + \cdots + (1+1-\frac{1}{2}) + 1+1-\frac{1}{2}\Big) \\
&= \frac{(3m+3)(3m+4)}{2} + 2\frac{3}{2}(m+1) \\
&= \frac{(3m+2)(3m+3)}{2} + 3(m+1)
\end{aligned}
$$

Since $y_{3m+2} = 3m+2,\ y_{3m+3} = 3m+4$ and $y_{3m+4} = 3(m+1)$,

$$
\begin{aligned}
x_{3m+2} - y_{3m+2} &= \Big(\frac{(3m+1)(3m+2)}{2} + 3m+2\Big) - (3m+2) &= \frac{(3m+1)(3m+2)}{2} \\
x_{3m+3} - y_{3m+3} &= \Big(\frac{(3m+2)(3m+3)}{2} + 3m+4\Big) - (3m+4) &= \frac{(3m+2)(3m+3)}{2} \\
x_{3m+4} - y_{3m+4} &= \Big(\frac{(3m+3)(3m+4)}{2} + 3(m+1)\Big) - 3(m+1) &= \frac{(3m+3)(3m+4)}{2}
\end{aligned}
$$

And $x_1 - y_1 = 0 - 0 = \frac{(1-1)\cdot 1}{2}$

Hence for $n = 1,\ 2,\ 3,\ \cdots$,

$$
x_n - y_n = \frac{n(n-1)}{2}
$$