Kyushu University

[1]

Let n be a positive integer. and let I_n be minimum value of the definite integral

$$\int_0^1 (\sin(2n\pi t) - xt - y)^2 \, dt$$

Find the limit $\lim_{n\to\infty} I_n$.

[2]

Given that two non-zero polynomials $f(x)$ and $g(x)$ which satisfy the equalities

$$f(x^2) = (x^2 + 2)g(x) + 7$$

$$g(x^3) = x^4 f(x) - 3x^2 f(x) - 6x^2 - 2$$

for any real number x.

(1) Show that the degree of $f(x)$ and the degree of $g(x)$ are both less than or equal to 2.

(2) Find $f(x)$ and $g(x)$.

[3]

Rolling a die three times, and let a, b and c are the numbers of die for the first, the second and the third rolling respectively.
Given that a quadratic equation

$$ax^2 + bx + c = 0$$

whose roots are z_1 and z_2.
And let $P_1(z_1)$ and $P_2(z_2)$ are points on the Argand diagram represented by z_1 and z_2 respectively.

(1) Find the probability that P_1 and P_2 are the same point.

(2) Find the probability that P_1 and P_2 are both on the unit circle.

(3) Let l_1 be the line passing through two points P_1 and O and let l_2 be the line passing through two points $P-2$ and O. Find the probability that the acute angle between two lines l_1 and l_2 is $60°$.

[4]

Given that three points $O(0,0)$, $A(2,0)$ and $B(1,\sqrt{3})$ in the xy-plane. Let P_1 be a point on the segment AB, different from A and B.
We define points P_2, P_3, $\cdots$ on the segment AB as:
When a point P_n is fixed, let Q_n be a point on the segment OB such that $P_nQ_n \perp OB$, and let R_n be a point on the segment OA such that $Q_nR_n \perp OA$ then P_{n+1} is defined as a point on the segment AB such that $R_nP_{n+1} \perp AB$.
Find the coordinates of the limit point P_n as $n \to \infty$.

[5]

Let a and b be complex numbers and let c be a complex number but not a pure complex number. Let C be a loci of point w which is defined as

$$w = \frac{az + b}{cz + 1}$$

where z moves on the imaginary axis, and satisfy the following three conditions.

(a) When $z = i$, $w = -i$ and when $z = -i$, $w = -i$.

(b) C is on the unit circle.

(c) The point -1 is not on C.

Find a, b and c. And find the equation of C and sketch it on the Argand diagram.