Matrices and Linear maps

Example 1

Calculate the following matrices.

(1) $3\begin{pmatrix}1 & -3\\ 2 & -4\end{pmatrix} - 2\begin{pmatrix}5 & 6\\ 7 & 3\end{pmatrix}$　(2) $\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix}$

(3) $\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$

(1)
$$\begin{aligned}3\begin{pmatrix}1 & -3\\ 2 & -4\end{pmatrix} - 2\begin{pmatrix}5 & 6\\ 7 & 3\end{pmatrix} &= \begin{pmatrix}3 & -9\\ 6 & -12\end{pmatrix} + \begin{pmatrix}-10 & -12\\ -14 & -6\end{pmatrix}\\ &= \begin{pmatrix}3-10 & -9-12\\ 6-14 & -12-6\end{pmatrix}\\ &= \begin{pmatrix}-7 & -19\\ -8 & -18\end{pmatrix}\end{aligned}$$

(2)
$$\begin{aligned}\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix} &= \begin{pmatrix}1\times 1 + 2\times 2\\ 3\times 1 + 1\times 2\end{pmatrix}\\ &= \begin{pmatrix}5\\ 5\end{pmatrix}\end{aligned}$$

(3)
$$\begin{aligned}\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix} &= \begin{pmatrix}1\times 4 + 3\times 3 & 1\times 1 + 3\times 2\\ 2\times 4 + 4\times 3 & 2\times 1 + 4\times 2\end{pmatrix}\\ &= \begin{pmatrix}13 & 7\\ 20 & 10\end{pmatrix}\end{aligned}$$

[1] Calculate the following matrices.

(1) $2\begin{pmatrix}1 & -2 & -3\\ 5 & 3 & 2\\ 2 & 0 & -1\end{pmatrix} - 3\begin{pmatrix}4 & 2 & -3\\ 5 & 3 & -2\\ 3 & 4 & 1\end{pmatrix}$　(2) $\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}$

(3) $\begin{pmatrix}1 & 0 & 2\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ 2 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$　(4) $\begin{pmatrix}1 & 0 & 0\\ -2 & -1 & 0\\ 0 & 1 & 1\end{pmatrix}^2$

Example 2

Find the inverse matrix of the following matrices.

(1) $A = \begin{pmatrix} 4 & -1 \\ 5 & 2 \end{pmatrix}$ (2) $B = \begin{pmatrix} 1 & 5 & -4 \\ 2 & 10 & -9 \\ -1 & -2 & 2 \end{pmatrix}$

$$\begin{aligned}(1)\ A^{-1} &= \frac{1}{detA}\begin{pmatrix} 2 & 1 \\ -5 & 4 \end{pmatrix} \\ &= \frac{1}{4 \times 2 - (-1) \times 5}\begin{pmatrix} 2 & 1 \\ -5 & 4 \end{pmatrix} \\ &= \frac{1}{13}\begin{pmatrix} 2 & 1 \\ -5 & 4 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}(2)\ B^{-1} &= \frac{1}{detB}\begin{pmatrix} 2 & -2 & -5 \\ 5 & -2 & 1 \\ 6 & -3 & 0 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 2 & -2 & -5 \\ 5 & -2 & 1 \\ 6 & -3 & 0 \end{pmatrix}\end{aligned}$$

[2] Find the inverse matrix of the following matrices.

(1) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (2) $B\begin{pmatrix} 1 & -7 \\ 3 & 4 \end{pmatrix}$

(3) $C = \begin{pmatrix} 1 & -1 & -2 \\ 2 & -1 & 5 \\ 1 & -1 & -1 \end{pmatrix}$ (4) $D = \begin{pmatrix} -2 & -3 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$

Example 3

Find the eigenvalues and the eigenvectors of the following matrices.

(1) $A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$ (2) $B = \begin{pmatrix} -5 & -8 & 6 \\ 8 & 9 & -4 \\ 7 & 8 & -4 \end{pmatrix}$

(1) The charasteristic equation is

$det(A - \lambda E) = 0$

$det\begin{pmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{pmatrix} = 0, \quad (1-\lambda)(4-\lambda) + 2 = 0, \quad \lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0, \quad \lambda = 2, 3$

Then the eigenvalues of A are $\lambda = 2,\ 3$.

The eigenvector belonging to $\lambda = 2$ is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

The eigenvector belonging to $\lambda = 3$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(2) The charasteristic equation is

$det(B - \lambda E) = 0$

$det\begin{pmatrix} -5-\lambda & -8 & 6 \\ 8 & 9-\lambda & -4 \\ 7 & 8 & -4-\lambda \end{pmatrix} = 0, \quad -(\lambda - 1)(\lambda - 2)(\lambda + 3) = 0$

Hence the eigenvalues are $\lambda = 1,\ 2,\ -3$

The eigenvector belonging to $\lambda = 1$ is $\begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$.

The eigenvector belonging to $\lambda = 2$ is $\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$.

The eigenvector belonging to $\lambda = -3$ is $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

[3] Find the eigenvalues and the eigenvectors of the following matrices.

(1) $A = \begin{pmatrix} 8 & 1 \\ 4 & 5 \end{pmatrix}$ (2) $B = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

(3) $C = \begin{pmatrix} 17 & -8 & 16 \\ 0 & 1 & 2 \\ -20 & 10 & -18 \end{pmatrix}$ (4) $D = \begin{pmatrix} -13 & 6 & -12 \\ -22 & 10 & -18 \\ 4 & -2 & 5 \end{pmatrix}$

Example 4

Find the eigenvalues and the eigenvectors, and diagonalise the following matrices.

(1) $A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$ (2) $B = \begin{pmatrix} 1 & 10 & -2 \\ 0 & -2 & 0 \\ 4 & 14 & -5 \end{pmatrix}$

(1) The charasteristic equation is

$det(A - \lambda E) = 0$

$det\begin{pmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{pmatrix} = 0, \quad (1-\lambda)(4-\lambda) + 2 = 0, \quad \lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0, \quad \lambda = 2, 3$

Then the eigenvalues of A are $\lambda = 2,\ 3$.

The eigenvector belonging to $\lambda = 2$ is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

The eigenvector belonging to $\lambda = 3$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Let $P = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$, then

$$P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

(2) The charasteristic equation is

$det(B - \lambda E) = -(\lambda + 1)(\lambda + 2)(\lambda + 3)$

Then the eigenvalues of B are $\lambda = -1,\ -2,\ -3$

Eigenvectors belonging to each eigenvalue are

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Let $P = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}$, then $P^{-1}BP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

[4] Find the eigenvalues and the eigenvectors, and diagonalise the following matrices.

(1) $A = \begin{pmatrix} 8 & 1 \\ 4 & 5 \end{pmatrix}$ (2) $B = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

(3) $C = \begin{pmatrix} 17 & -8 & 16 \\ 0 & 1 & 2 \\ -20 & 10 & -18 \end{pmatrix}$ (4) $D = \begin{pmatrix} -4 & 4 & -2 \\ -4 & 6 & -4 \\ 16 & -8 & 2 \end{pmatrix}$

Example 5

Given $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.
Find $A^3 - 6A^2 + 4A + 5E$

From the Cayley-Hamilton theorem, we have

$$A^2 - 4A - 5E = O$$

Then

$$A^3 - 6A^2 + 4A + 5E = (A^2 - 4A - 5E)(A - 2E) + A - 5E = A - 5E = \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}$$

[5] Given $A = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}$. Evaluate $A^2,\ A^3,\ A^4,\ A^{10}$

[6]