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[1]

Given that

$$I_n = \int_0^{\frac{\pi}{3}} \frac{d\theta}{\cos^n \theta}$$

where n is a positive integer.

(1) Find I_1.
You may use the result that $\dfrac{1}{\cos\theta} = \dfrac{1}{2}\left(\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{\cos\theta}{1-\sin\theta}\right)$.

(2) Assume that $n \geq 3$. Express I_n with I_{n-2} and n.

(3) On the xy-plane, let D be a disc whose centre is the origin O and that the radius is 1. Let C be a cone whose vertex is a point $(0, 0, 1)$ and that the base is D. Let S be a smaller part when C is cut by a plane $x = \dfrac{1}{2}$. Find the volume of S, considering the section of S by the planes parallel to the z-axis.

[2]

Given that an equilateral right triangle ABC, where $\angle BAC = \dfrac{\pi}{2}$ and given that a plane P.
The point A is on the plane P and points B and C are not on the plane P. The plane P divides the space for two parts and the points B and C are in the same part. Ler B' and C' be two points on the plane P such that $BB' \perp P$ and $CC' \perp P$.

(1) Show that $\overrightarrow{AB'} \cdot \overrightarrow{AC'} + \overrightarrow{B'B} \cdot \overrightarrow{C'C} = \overrightarrow{0}$

(2) Show that $\angle B'AC' > \dfrac{\pi}{2}$.

(3) The length of the sides of the triangle $AB'C'$ on the plane P are 4, $\sqrt{21}$ and 7. Find the length of AB.

[3]

Let n be a positive integer and $\sqrt{n}$ not be an integer.
When we represent $\sqrt{n}$ to a decimal system, the first decimal place is 0 and the second decimal place is not 0.

(1) Find the smallest n.

(2) When we order such integer n from the smallest to the largest, find the 10th smallest integer$n$.

[4]

The number of permutation of n positive integers $1,\ 2,\ \cdots,\ n$ is $n!$. Let $(a_1,\ a_2,\ \cdots,\ a_n)$ be one of this permutation. When $a_i = j$ and $a_j = k$, we shall write down as $a_i = j \to a_j = k$. We continue this like $a_i = j \to a_j = k \to a_k = l \to \cdots$. For example, given that a permutation $(a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6, a_7) = (2,\ 5,\ 6,\ 1,\ 4,\ 3,\ 7)$.

(i) $a_1 = 2 \to a_2 = 5 \to a_5 = 4 \to a_4 = 1 \to a_1 = 2$

(ii) $a_3 = 6 \to a_6 = 3 \to a_3 = 6$

(iii) $a_7 = 7 \to a_7 = 7$

Then when we start a_i of any index i, we will come back to the same a_i. We call this sequence the cycle and the number of different integers a_i in this cycle is called the length of the cycle.
For the former example, the length of the cycle (i) is 4, the length of the cycle (ii) is 2 and the length of the cycle (iii) is 1.

(1) Let $n = 3$. Find the probability that a random chosen permutation $(a_1,\ a_2,\ \cdots,\ a_n)$ has at least one cycle whose length is 1.

(2) Let $n = 4$. Find all permutations which has at least one cycle whose length is 4.

(3) Show that, for $k \leq n$,

$$\sum_{j=k}^{n} \frac{1}{j} > \log(n+1) - \log k$$

(4) Assume that n is an odd number. Show that $p > \log 2$, where p is a probability that a chosen permutation has a cycle whose length is larger than or equal to $\dfrac{n+1}{2}$.