# The First Order Linear Ordinary Differential Equations

$$y' + P(x)y = Q(x)$$

**I. When** $Q(x) = 0$

Our equation is as

$$y' + P(x)y = 0$$

$$\frac{dy}{dx} = -P(x)y$$

Then

$$\frac{dy}{y} = -P(x)dx$$

which is a separable differential equation.
Integrate both sides of the equation;

$$\int \frac{dy}{y} = -\int P(x)dx$$

Then

$$\ln|y| = -\int P(x)dx$$

Let $F(x) = \int P(x)dx$, then

$$\ln|y| = -F(x) + C_1$$

$$|y| = e^{-F(x)+C_1}$$

$$y = \pm e^{C_1} e^{-F(x)}$$

$$y = Ce^{-F(x)} \qquad \cdots \quad (*)$$

**II. When** $Q(x) \neq 0$

Suppose that $C = H(x)$ in $(*)$, then

$$y = H(x)e^{-F(x)}$$

When we can find the function $H(x)$, we can determine our function $y$.

$$H(x) = e^{F(x)}y$$

Differentiate this:

$$H'(x) = e^{F(x)}y' + e^{F(x)}F'(x)y$$

Since $F'(x) = P(x)$,

$$H'(x) = e^{F(x)}y' + e^{F(x)}P(x)y = e^{F(x)}(y' + P(x)y)$$

From our differential equation $y' + P(x)y = Q(x)$,

$$H'(x) = e^{F(x)}Q(x)$$

Hence

$$H(x) = \int e^{F(x)}Q(x)\,dx$$

And therefore

$$y = e^{-F(x)} \int e^{F(x)}Q(x)\,dx$$

## III. Summary

For obtaining $H'(x)$, we multiply the both sides of the differential equation by $e^{F(x)}$, which is called **the integrating factor**.

And integrating both sides, we can find the function $H(x)$,
hence our solution $y$ of the differential equation:

$$y' + P(x)y = Q(x)$$

Let $F(x) = \int P(x)\,dx$, then the integrating factor is $e^{F(x)}$.
Multiply both sides of the differential equation by this integrating factor,

$$e^{F(x)}y' + e^{F(x)}P(x)y = e^{F(x)}Q(x)$$

$$e^{F(x)}y' + (e^{F(x)})'y = e^{F(x)}Q(x)$$

Then

$$\frac{d}{dx}(e^{F(x)}y) = e^{F(x)}Q(x)$$

Integrating both sides of this equation,

$$e^{F(x)}y = \int e^{F(x)}Q(x)\,dx$$

Hence

$$y = e^{-F(x)} \int e^{F(x)} Q(x)\, dx$$

## Example

Find the general solution of

$$y' + 2y = 5 \sin x$$

$$F(x) = \int 2dx = 2x$$

Then the integrating factor is $e^{2x}$ Multiply both sides of the differential equation by this integrating factor,

$$e^{2x} y' + 2ye^{2x} = 5e^{2x} \sin x$$

$$\frac{d}{dx}(e^{2x} y) = 5e^{2x} \sin x$$

Integrating both sides of this equation,

$$e^{2x} y = \int 5e^{2x} \sin x\, dx$$

Let $H(x) = \int 5e^{2x} \sin x\, dx$, then integrating by parts (twice),

$$\begin{aligned}
H(x) &= \int e^{2x}(5\sin x)dx \\
&= \frac{1}{2}e^{2x}(5\sin x) - \int \frac{1}{2}e^{2x}(5\cos x)dx \\
&= \frac{5}{2}e^{2x}\sin x - \frac{5}{2}\Big(\frac{1}{2}e^{2x}\cos x - \int \frac{1}{2}e^{2x}(-\sin x)dx\Big) \\
&= \frac{5}{4}e^{2x}(2\sin x - \cos x) - \frac{1}{4}\int e^{2x}(5\sin x)dx \\
&= \frac{5}{4}e^{2x}(2\sin x - \cos x) - \frac{1}{4}H(x)
\end{aligned}$$

Then

$$\frac{5}{4}H(x) = \frac{5}{4}e^{2x}(2\sin x - \cos x) + C_1$$

Therefore

$$H(x) = e^{2x}(2\sin x - \cos x) + C$$

Hence

$$e^{2x}y = e^{2x}(2\sin x - \cos x) + C$$

$$y = -\cos x + 2\sin x + Ce^{-2x}$$

Exercise

[1] Solve the following ODE of order 1.

(i) $y' + (\tan x)y = x \cos x$

(ii) $y' + \dfrac{2x}{x^2+1} y = 4x$

(iii) $y' + 3xy = 4x$

(iv) $y' = x + y$

(v) $xy' + y = x \sin x \quad (x > 0)$

[2] The differential equation

$$3xy^2 \frac{dy}{dx} + 2y^3 = \frac{\cos x}{x}$$

is to be solved for $x > 0$. Use the substitution $u = y^3$ to find the general solution for $y$ in terms of $x$.

[3] (i) Find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = \sin 2x$$

expressing $y$ in terms of $x$ in your answer.

In a particular case, it is given that $y = \dfrac{2}{\pi}$ when $x = \dfrac{1}{4}\pi$.

(ii) Find the solution of the differential equation in this case.

(iii) Write down a function to which $y$ approximates when $x$ is large and positive.

[4] (i) Show that the substitution$z = y^{-2}$ transforms the differential equation

$$\frac{dy}{dx} + 2xy = xe^{-x^2}y^3 \qquad \cdots (*)$$

into the differential equation

$$\frac{dz}{dx} - 4xz = -2xe^{-x^2} \qquad \cdots (**)$$

(ii) Solve differential equation (**) to find $z$ as a function of $x$.

(iii) Hence find the general solution of differential equation (*), giving your answer in the form $y^2 = f(x)$.

[5] (i) Show that the substitution$v = y^{-3}$ transforms the differential equation

$$x\frac{dy}{dx} + y = Ax^4y^4 \qquad \cdots (*)$$

into the differential equation

$$\frac{dv}{dx} - \frac{3v}{x} = -6x^3 \qquad \cdots (**)$$

(ii) By solving differential equation (**), find the general solution of differential equation (*) in the form $y^3 = f(x)$.

[6] Given that $P(x) = Q(x)R'(x) - Q'(x)R(x)$, write down an expression for

$$\int \frac{P(x)}{(Q(x))^2}\,dx$$

(i) By choosing the function $R(x)$ to be of the form $a + bx + cx^2$, nd

$$\int \frac{5x^2 - 4x - 3}{(1 + 2x + 3x^2)^2}\,dx$$

Show that the choice of $R(x)$ is not unique and, by comparing the two functions $R(x)$ corresponding to two different values of $a$ , explain how the different choices are related.

(ii) Find the general solution of

$$(1+\cos x+2\sin x)\frac{dy}{dx}+(\sin x-2\cos x)y = 5-3\cos x+4\sin x$$