The 2nd order Linear Ordinary Differential Equations with constant coefficients

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R(x)$$

I. When $R(x) = 0$: homogeneous

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad \cdots (1)$$

The solution of this equation (1) is called the complementary function and written as

$$y_{CF} = y_1 + y_2$$

where y_1 and y_2 are independent functions.
For finding y_1 and y_2, suppose that $y = Ae^{mx}$, then

$$\frac{dy}{dx} = mAe^{mx}, \quad \frac{d^2y}{dx^2} = m^2Ae^{mx}$$

Substitute to (1)

$$m^2Ae^{mx} + PmAe^{mx} + QAe^{mx} = 0$$

$$Ae^{mx}(m^2 + Pm + Q) = 0$$

Hence

$$m^2 + Pm + Q = 0 \quad \cdots (2)$$

which is called the **auxiliary equation** of (1).

1) When the auxiliary equation (2) has two different roots $m = m_1,\ m = m_2$
The independent functions are

$$y_1 = Ae^{m_1x} \quad \text{and} \quad y_2 = Be^{m_2x}$$

Then the complementary function is

$$\boldsymbol{y_{CF} = y_1 + y_2 = Ae^{m_1x} + Be^{m_2x}}$$

2) When the auxiliary equation (2) has one root (double roots) $m = m_1$
The independent functions are

$$y_1 = Ae^{m_1x} \quad \text{and} \quad y_2 = Bxe^{m_1x}$$

Then the complementary function is

$$\boldsymbol{y_{CF} = y_1 + y_2 = Ae^{m_1 x} + Bxe^{m_1 x}}$$

When the auxiliary equation (2) has two imaginary roots $m = a + ib$ and $m = a - ib$, we may write down as

$$\begin{aligned} y_{CF} &= y_1 + y_2 \\ &= C_1 e^{(a+ib)x} + C_2 e^{(a-ib)x} \\ &= C_1 e^{ax} e^{ibx} + C_2 e^{ax} e^{-ibx} \\ &= e^{ax}(C_1 e^{ibx} + C_2 e^{-ibx}) \\ &= e^{ax}(C_1(\cos bx + i \sin bx) + C_2(\cos bx - i \sin bx)) \\ &= e^{ax}((C_1 + C_2)\cos bx + i(C_1 - C_2)\sin bx) \\ &= e^{ax}(A \cos bx + B \sin bx) \end{aligned}$$

$$\boldsymbol{y_{CF} = y_1 + y_2 = e^{ax}(A \cos bx + B \sin bx)}$$

II. When $R(x) \neq 0$: inhomogeneous

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R(x)$$

The general solution of this equation is written as

$$y = y_{CF} + y_{PI}$$

where y_{CF} is the complementary function of the homogeneous equation and y_{PI} is the particular integral.
For finding the particular integral y_{PI}, we shall try

$R(x)$	y_{PI}
$R(x) = k$	$y_{PI} = C$
$R(x) = ax + b$	$y_{PI} = Cx + D$
$R(x) = ax^2 + bx + c$	$y_{PI} = Cx^2 + Dx + E$
$R(x) = k\sin ax$ or $R(x) = k\cos ax$	$y_{PI} = C\cos ax + D\sin ax$
$R(x) = ke^{ax}$	$y_{PI} = Ce^{ax}$

Example
Find the particular integral of

$$y'' + 3y' + 2y = 3\sin 2x$$

Suppose that $y = C\cos 2x + D\sin 2x$, then

$$y' = -2C\sin 2x + 2D\cos 2x$$

$$y'' = -4C\cos 2x - 4D\sin 2x$$

Then

$$(-4C\cos 2x - 4D\sin 2x) + 3(-2C\sin 2x + 2D\cos 2x) + 2(C\cos 2x + D\sin 2x) = 3\sin 2x$$

$$(-2C + 6D)\cos 2x + (-6C - 2D)\sin 2x = 3\sin 2x$$

Comparing the coefficients.

$$-2C + 6D = 0, \quad -6C - 2D = 3$$

$$C = -\frac{9}{20}, \quad D = -\frac{3}{20}$$

Hence the particular integral is

$$y_{PI} = -\frac{9}{20}\cos 2x - \frac{3}{20}\sin 2x$$

Example

Fine the general solution of

$$y'' + 3y' + 2y = e^{-x}$$

Solving the auxiliary equation

$$m^2 + 3m + 2 = 0$$

we have

$$(m+1)(m+2) = 0$$

$$m = -1 \quad \text{or} \quad m = -2$$

Then the complimentary function is

$$y_{CF} = Ae^{-x} + Be^{-2x}$$

Since Ae^{-x} is included in the C.F. we cannot use $y = Ce^{-x}$ as the particular integral.

Then we suppose that $y_{PI} = Cxe^{-x}$.

$$y = Cxe^{-x}, \quad y' = Ce^{-x} - Cxe^{-x}, \quad y'' = -Ce^{-x} - (Ce^{-x} - Cxe^{-x}) = -2Ce^{-x} + Cxe^{-x}$$

Then

$$(-2Ce^{-x} + Cxe^{-x}) + 3(Ce^{-x} - Cxe^{-x}) + 2Cxe^{-x} = e^{-x}$$

$$Ce^{-x} = e^{-x}$$

Comparing the coefficient, we have

$$C = 1$$

Hence the particular integral is

$$y_{PI} = xe^{-x}$$

Hence the general solution is

$$y = y_{CF} + y_{PI} = Ae^{-x} + Be^{-2x} + xe^{-x}$$

Exercise

[1] Find the general solution of the following differential equations.

(i) $y'' - 2y' - 3y = 6$

(ii) $y'' + 5y' + 6y = 2x$

(iii) $y'' + 2y' + y = \cos 3x$

(iv) $y'' + 4y' + 5y = 2e^{-2x}$

[2] (i) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 27e^{-x}$$

(ii) Find the particular solution that satisfies $y = 0$ and $\dfrac{dy}{dx} = 0$ when $x = 0$.

[3] (i) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2\sin x \qquad \cdots (*)$$

(ii) Given that $y = 0$ and $\dfrac{dy}{dx} = 1$ when $x = 0$.
Find the particular solution of differential equation (*).

[4] (i) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 27z^{-x} \qquad \cdots (*)$$

(ii) Given that $y = 0$ and $\dfrac{dy}{dx} = 0$ when $x = 0$.
Find the particular solution of differential equation (*).

[5] Show that, if $y = e^x$, then

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0 \qquad \cdots (*)$$

In order to find other solution of this differential equation, now let $y = ue^x$, where u is a function of x. By substituting this into (*), show that

$$(x-1)\frac{d^2u}{dx^2} + (x-2)\frac{du}{dx} = 0 \qquad \cdots (**)$$

By setting $\dfrac{du}{dx} = v$ in (**) and solving the resulting first order differential equation for v, find u in termsof x. Hence show that $y = Ax + Be^x$ satisfies (*), where A and B are any constants.

[6] (i) Find functions $a(x)$ and $b(x)$ such that $u = x$ and $u = e^{-x}$ both satisfy the equation,

$$\frac{d^2u}{dx^2} + a(x)\frac{du}{dx} + b(x)u = 0$$

For these functions $a(x)$ and $b(x)$, write down the general solution of the equation.

Show that the substitution $y = \dfrac{1}{3u}\dfrac{du}{dx}$ transforms the equation

$$\frac{dy}{dx} + 3y^2 + \frac{x}{1+x}y = \frac{1}{3(1+x)} \qquad \cdots (*)$$

into

$$\frac{d^2u}{dx^2} + \frac{x}{1+x}\frac{du}{dx} - \frac{1}{1+x} = 0$$

and hence show that the solution of equation (*) that satisfies $y = 0$ at $x = 0$ is given by $y = \dfrac{1-e^{-x}}{3(x+e^{-x})}$.

(ii) Find the solution of the equation

$$\frac{dy}{dx} + y^2 + \frac{x}{1-x}y = \frac{1}{1-x}$$

that satisfies $y = 2$ at $x = 0$.