Entrance Exams of Osaka University 2022

[1]

Let r be a positive real number. In the complex plane, the locus of z is the circle whose centre is the point $\frac{3}{2}$ and the radius is r. Find the locus w such that

$$z + w = zw$$

The equation of z is

$$|z - \frac{3}{2}| = r \cdots (*)$$

From the equation $z + w = zw$,

$$z = \frac{w}{w-1}$$

Substitute this in (*).

$$\left|\frac{w}{w-1} - \frac{3}{2}\right| = r$$

$$\left|\frac{2w - 3(w-1)}{2(w-1)}\right| = r$$

$$\left|\frac{-w+3)}{2(w-1)}\right| = r$$

$$|-w+3| = 2r|w-1| \qquad or \qquad |w-3| = 2r|w-1|$$

When $r = \frac{1}{2}$,

$$|w-3| = |w-1|$$

The locus of w is the perpendicular bisector of the segment joining the point 1 and the point 3.

When $r \neq \frac{1}{2}$,

$$|w-3|^2 = 4r^2|w-1|^2$$

$$(w-3)(\overline{w}-3) = 4r^2(w-1)(\overline{w}-1)$$

$$w\overline{w} - 3w - 3\overline{w} + 9 = 4r^2(w\overline{w} - w - \overline{w} + 1)$$

$$(4r^2-1)w\overline{w}-(4r^2-3)w-(4r^2-3)\overline{w}+4r^2-9=0$$

$$w\overline{w}-\frac{4r^2-3}{4r^2-1}w-\frac{4r^2-3}{4r^2-1}\overline{w}+\frac{4r^2-9}{4r^2-1}=0$$

$$w\left(\overline{w}-\frac{4r^2-3}{4r^2-1}\right)-\frac{4r^2-3}{4r^2-1}\overline{w}+\frac{4r^2-9}{4r^2-1}=0$$

$$w\left(\overline{w}-\frac{4r^2-3}{4r^2-1}\right)-\frac{4r^2-3}{4r^2-1}\left(\overline{w}-\frac{4r^2-3}{4r^2-1}\right)-\left(\frac{4r^2-3}{4r^2-1}\right)^2+\frac{4r^2-9}{4r^2-1}=0$$

$$\left(w-\frac{4r^2-3}{4r^2-1}\right)\left(\overline{w}-\frac{4r^2-3}{4r^2-1}\right)=\frac{16r^2}{(4r^2-1)^2}$$

$$\left|w-\frac{4r^2-3}{4r^2-1}\right|^2=\frac{16r^2}{(4r^2-1)^2}$$

$$\left|w-\frac{4r^2-3}{4r^2-1}\right|=\frac{4r}{|4r^2-1|}$$

The locus of w is a circle whose centre is the point $\frac{4r^2-3}{4r^2-1}$ and the radius is $\frac{4r}{|4r^2-1|}$.

Conclusion:
The locus of w is

- When $r=\frac{1}{2}$, the perpendicular bisector of the segment joining 1 and 3.
- When $r\neq\frac{1}{2}$, the circle whose centre is the point $\frac{4r^2-3}{4r^2-1}$ and the radius is $\frac{4r}{|4r^2-1|}$.

[2]

Let $\alpha = \dfrac{2\pi}{7}$.

(1) Show that $\cos 4\alpha = \cos 3\alpha$.

(2) Given that $f(x) = 8x^3 + 4x^2 - 4x - 1$. Show that $f(\cos\alpha) = 0$.

(3) Show that $\cos\alpha$ is an irrational number.

(1)

$$\cos 4\alpha = \cos\frac{8\pi}{7} = \cos(\pi + \frac{\pi}{7}) = -\cos\frac{\pi}{7}$$

$$\cos 3\alpha = \cos\frac{6\pi}{7} = \cos(\pi - \frac{\pi}{7}) = -\cos\frac{\pi}{7}$$

Hence $\cos 4\alpha = \cos 3\alpha$.

(2) From $\cos 4\alpha = \cos 3\alpha$,

$$2\cos^2 2\alpha - 1 = 4\cos^3\alpha - 3\cos\alpha$$

$$2(2\cos^2\alpha - 1)^2 - 1 = 4\cos^3\alpha - 3\cos\alpha$$

$$8\cos^4\alpha - 4\cos^3\alpha - 8\cos^2\alpha + 3\cos\alpha + 1 = 0$$

$$(\cos\alpha - 1)(8\cos^3\alpha + 4\cos^2\alpha - 4\cos\alpha - 1) = 0$$

Since $\cos\alpha = \cos\dfrac{2\pi}{7} \neq 1$, $\cos\alpha - 1 \neq 0$. Then

$$8\cos^3\alpha + 4\cos^2\alpha - 4\cos\alpha - 1 = 0$$

i.e.

$$f(\cos\alpha) = 0$$

(3) $f(x) = 8x^3 + 4x^2 - 4x - 1$,

$$f'(x) = 24x^2 + 8x - 4 = 4(6x^2 + 2x - 1)$$

When $f'(x) = 0$,

$$6x^2 + 2x - 1 = 0, \qquad \text{then} \qquad x = \frac{-1 \pm \sqrt{7}}{6}$$

$f(0) = -1$ and $f(1) = 7$, we have the variation table of f in the interval $0 \leq x \leq 1$ as

x	0		$\frac{-1+\sqrt{7}}{6}$		1
$f'(x)$		$-$	0	$+$	
$f(x)$	-1	$\searrow$	(minimum)	$\nearrow$	7

Since

$$f(\frac{\sqrt{3}}{2}) = 8(\frac{\sqrt{3}}{2})^3 + 4(\frac{\sqrt{3}}{2})^2 - -4(\frac{\sqrt{3}}{2}) - 1 = 3\sqrt{3} + 3 - 2\sqrt{2} - 1 = \sqrt{3} + 2 > 0$$

Then we can write our variation table of f as

x	0		$\frac{-1+\sqrt{7}}{6}$		$\frac{\sqrt{3}}{2}$		1
$f'(x)$		$-$	0	$+$	$+$	$+$	
$f(x)$	-1	$\searrow$	(minimum)	$\nearrow$	$\sqrt{4}+2$	$\nearrow$	7

From this table we can say that in the interval $0 \leq x \leq 1$, the equation $f(x) = 0$ has only one real solution, which is larger than $\frac{-1+\sqrt{7}}{6}$ and smaller than $\frac{\sqrt{3}}{2}$.

As

$$\frac{\pi}{3} = \frac{7\pi}{21} > \frac{6\pi}{21} = \frac{2\pi}{7} = \alpha$$

$$\cos\frac{\pi}{3} < \cos\alpha$$

Then

$$\frac{\sqrt{3}}{2} < \cos\alpha < 1$$

It means that $x = \cos\alpha$ is not a real solution of $f(x) = 0$.
Since $x = \cos\alpha$ is a solution of $f(x) = 0$,
$\cos\alpha$ must be an irrational number.

[3]

Let t be a positive real number and given that two points $P(0,t)$ and $Q(\frac{1}{t},0)$ in the plane. When $1 \leq t \leq 2$, draw the region where the segment PQ may pass.

The equation of the segment PQ is

$$y = -t^2x + t \qquad (x \geq 0,\ y \geq 0)$$

We consider this equation as a quadratic equation for t,

$$xt^2 - t + y = 0 \qquad \cdots (*)$$

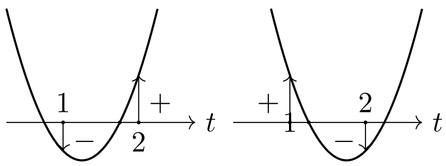
The coordinates (x,y) in the region, where the segment PQ may pass, are satisfy the condition such that the quadratic equation (*) has at least one real root in the interval $1 \leq t \leq 2$.

i) When $x = 0$, the equation is $-t + y = 0$.
Then $1 \leq y \leq 2$.

ii) When $x \neq 0$,
let $f(t) = xt^2 - t + y$.

$$f(t) = x(t - \frac{1}{2x})^2 - \frac{1}{4x} + y$$

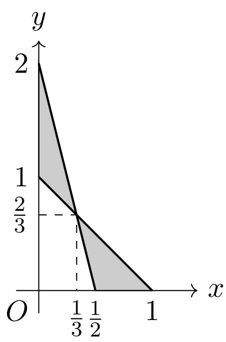
ii) - (a) Condition for the eqution (*) has two real roots and one of the roots is in the interval $1 \leq t \leq 2$ and other root is outside of this interval (or both 1 and 2 are the roots of (*))



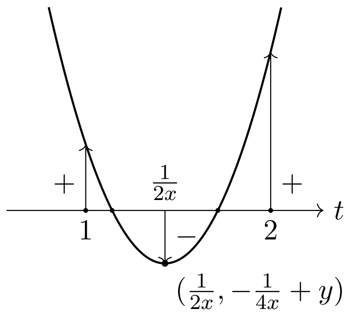
$$f(1)f(2) \leq 0$$

$$(x - 1 + y)(4x - 2 + y) \leq 0$$

$$(y \geq -x + 1 \quad \text{and} \quad y \leq -4x + 2) \quad \text{or} \quad (y \leq -x + 1 \quad \text{and} \quad y \geq -4x + 2)$$

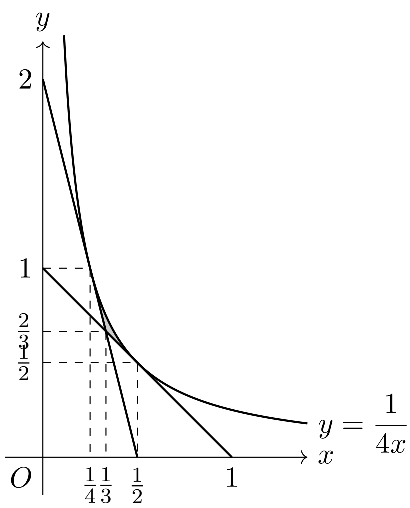


ii)-(b) Condition for the eqution (*) has two real roots and both roots are in the interval $1 \leq t \leq 2$,

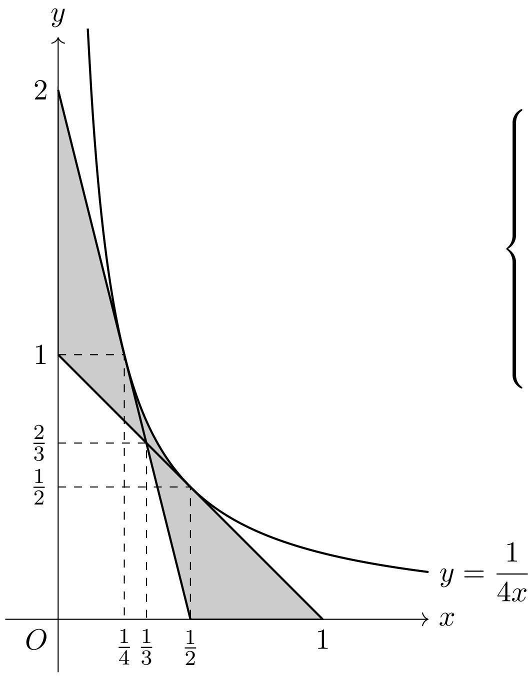


$$f(1) \geq 0 \quad \text{and} \quad f(2) \geq 0 \quad \text{and} \quad 1 \leq \frac{1}{2x} \leq 2 \quad \text{and} \quad -\frac{1}{4x} + y \leqq 0$$

$$y \geq -x+1 \quad \text{and} \quad y \geq -4x+2 \quad \text{and} \quad \frac{1}{4} \leq x \leq \frac{1}{2} \quad \text{and} \quad y \leqq \frac{1}{4x}$$



Hence the required region is the shaded part of the diagram below.



$$
\begin{cases}
-x+1 \leq y \leq -4x+2 & (0 \leq x \leq \frac{1}{4}) \\
-x+1 \leq y \leq \frac{1}{4x} & (\frac{1}{4} \leq x \leq \frac{1}{3}) \\
-4x+2 \leq y \leq \frac{1}{4x} & (\frac{1}{3} \leq x \leq \frac{1}{2}) \\
0 \leq y \leq -x+1 & (\frac{1}{2} \leq x \leq 1)
\end{cases}
$$

[4]

Given that $f(x) = \log(x+1) + 1$.

(1) Show that the equation $f(x) = x$ has one and only one solution for $x > 0$.

(2) Let α be the unique solution of (1). Show that

$$0 < \frac{\alpha - f(x)}{\alpha - x} < f'(x)$$

if a real number x satisfies $0 < x < \alpha$.

(3) The sequence $\{x_n\}$ is defined by

$$x_1 = 1, \quad x_{n+1} = f(x_n) \quad (n = 1,\ 2,\ 3,\ \cdots)$$

Show that, for any natural number n,

$$\alpha - x_{n+1} < \frac{1}{2}(\alpha - x_n)$$

(4) Show that $\lim_{n\to\infty} x_n = \alpha$

(1) Let $g(x) = f(x) - x = \log(x+1) + 1 - x$.

$$g'(x) = \frac{1}{x+1} - 1 = -\frac{x}{x+1}$$

Then, for $x > 0$, $g'(x) < 0$.
$g(x)$ is strictly decreasing for $x > 0$.

Since $g(0) = \log 1 + 1 - 0 = 1 > 0$ and
$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} (\log(x+1)+1-x) = \lim_{x\to\infty} x\left(\frac{\log(x+1)}{x} + \frac{1}{x} - 1\right) = \lim_{x\to\infty}(-x) = -\infty < 0$

Then $g(x) = 0$ has only one solution for $x > 0$.
Hence $f(x) = x$ has only one solution for $x > 0$.

(2) $f(x) = \log x + 1) + 1$

$$f'(x) = \frac{1}{x+1} > 0 \quad \text{for } x > 0$$

Then $f(x)$ is strictly increasing for $x > 0$.
Therefore $\alpha - f(x) = f(\alpha) - f(x) > 0$ for $0 < x < \alpha$.
And $\alpha - x > 0$, hence

$$0 < \frac{\alpha - f(x)}{\alpha - x}$$

According to the mean value theorem, there exists c such that $x < c < \alpha$ and

$$\frac{f(\alpha) - f(x)}{\alpha - x} = f'(c)$$

On the other hand,

$$f''(x) = -\frac{1}{(x+1)^2} < 0$$

Then $f'(x)$ is strictly decreasing.

Therefore $f'(c) < f'(x)$ for $0 < x < c < \alpha$.
Then

$$\frac{f(\alpha) - f(x)}{\alpha - x} = f'(c) < f'(x)$$

And $f(\alpha) = \alpha$, hence

$$\frac{\alpha - f(x)}{\alpha - x} < f'(x)$$

Then we proved that

$$0 < \frac{\alpha - f(x)}{\alpha - x} < f'(x)$$

(3) First we shall prove, by induction, that $1 \leq x_n < \alpha$ for any positive integer n.

We have proved that $g(x) = f(x) - x$ is strictly decreasing for $x > 0$ in the part of (1).

$$g(1) = f(1) - 1 = \log(2) + 1 - 1 = \log 2 > 0 \quad \text{and} \quad g(\alpha) = 0$$

Then $1 < \alpha$.

Therefore $1 \leq x_1 = 1 < \alpha$.

Suppose that $1 \leq x_n < \alpha$,

$$x_{n+1} = f(x_n) = \log(x_n + 1) + 1$$

Since $1 \leq x_n < \alpha$,

$$\log(1+1) + 1 \leq \log(x_n + 1) + 1 < \log(\alpha + 1) + 1$$

$$1 \leq \log 2 + 1 \leq ()x_n + 1) + 1 < \log(\alpha + 1) + 1 = \alpha$$

Hence $1 \leq x_{n+1} < \alpha$.
Therefore we have proved that, for all positive integers n, $1 \leq x_n < \alpha$.

Then from the part of (2),

$$0 < \frac{\alpha - f(x_n)}{\alpha - x_n} < f'(x_n)$$

$$0 < \alpha - f(x_n) < f'(x_n)(\alpha - x_n)$$

$$0 < \alpha - x_{n+1} < f'(x_n)(\alpha - x_n)$$

Since $f'(x) = \dfrac{1}{x+1} \leq \dfrac{1}{2} \quad \text{for} \quad x \geq 1,$

$$0 < \alpha - x_{n+1} < f'(x_n)(\alpha - x_n) \leq \frac{1}{2}(\alpha - x_n)$$

I.e. $\alpha - x_{n+1} < \dfrac{1}{2}(\alpha - x_n)$.

(4) From the result of the part (3),

$$0 < \alpha - x_n < \frac{1}{2}(\alpha - x_{n-1}) < \left(\frac{1}{2}\right)^2(\alpha - x_{n-2}) < \cdots < \left(\frac{1}{2}\right)^{n-1}(\alpha - x_1)$$

Then

$$0 \leq \lim_{n\to\infty}(\alpha - x_n) \leq \lim_{n\to\infty}\left(\frac{1}{2}\right)^{n-1}(\alpha - x_1) = 0$$

$$\lim_{n\to\infty}(\alpha - x_n) = 0$$

Hence

$$\lim_{n\to\infty} x_n = \alpha$$

[5]

Let C be the curve, which is defined by the parametric equations:

$$x = e^t \cos t + e^\pi, \quad y = e^t \sin t \quad (0 \leq t \leq \pi)$$

Find the area of the region surrounded by the curve C and the x-axis.

The curve C_0, defined by

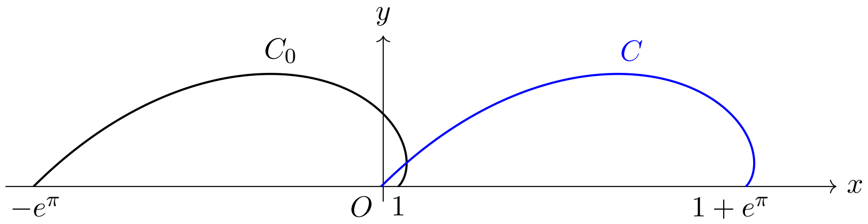
$$x = e^t \cos t, \qquad y = e^t \sin t$$

is obtained by translating the curve C through $-e^\pi$ units parallel to the x-axis.

$$r = \sqrt{x^2 + y^2} = \sqrt{e^{2t}\cos^2 t + e^{2t}\sin^2 t} = \sqrt{e^{2t}(\cos^2 t + \sin^2 t)} = \sqrt{e^{2t}} = e^t$$

Then the polar equation of the curve C_0 can be written as

$$r = e^\theta \qquad (0 \leq \theta \leq \pi)$$



The required region and the region surrounded by C_0 and the x-axis are congruent.

Hence the area of the region surrounded by the curve C and the x-axis is

$$\begin{aligned}
(\text{Area of the region}) &= \frac{1}{2}\int_0^\pi r^2 d\theta \\
&= \frac{1}{2}\int_0^\pi e^{2\theta} d\theta \\
&= \frac{1}{2}\left[\frac{1}{2}e^{2\theta}\right]_0^\pi \\
&= \frac{1}{4}(e^{2\pi} - 1)
\end{aligned}$$