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[1]

Let log be the natural logarithm whose base is e.

(1) Let b be a real number.
Show that the function

$$f(x) = \int_x^b e^{-\frac{t^2}{2}}\,dt - \frac{x}{x^2+1}e^{-\frac{x^2}{2}}$$

is decreasing.

(2) Show that, for positive real numbers a and b where $a \leq b$, the inequality

$$\frac{a}{a^2+1}e^{-\frac{a^2}{2}} - \frac{b}{b^2+1}e^{-\frac{b^2}{2}} \leq \int_a^b e^{-\frac{t^2}{2}}\,dt \leq e^{-\frac{a^2}{2}}(b-a)$$

(3) Let $\{I_n\}$ be a sequence defined by

$$I_n = \int_1^2 e^{-\frac{nt^2}{2}}\,dt \qquad (n = 1,\ 2,\ 3,\ \cdots)$$

Find the limit

$$\lim_{n\to\infty} \frac{1}{n}\log I_n$$

You may use the result that $\lim\limits_{n\to\infty} \frac{1}{n}\log(n+1) = 0$.

[2]

Given that

$$w = \cos\frac{A\pi}{3+b} + i\sin\frac{a\pi}{3+b}$$

where a and b are positive integers.
We define z_n as

$$z_1 = 1,\ z_2 = 1 - w,\ z_n = (1-w)z_{n-1} + wz_{n-2}\ (n = 3,\ 4,\ 5,\ \cdots)$$

(1) When $a = 4$ and $b = 3$, sketch the shape tracing points $z_1,\ z_2,\ z_3,\ z_4,\ z_5,\ z_6,\ z_7$ in this order in Argand diagram.

(2) When $a = 2$ and $b = 1$, find the value of z_{63}.

(3) Throw a die twice. Let a be the first times top face number and let b be the second one. Find the probability for $z_{63} = 0$.

[3]

Let s and t be real valuables satisfying the condition $s^2 + t^2 \leq 6$. Let A be a region of points whose coordinates are $(s + t, st)$ on the xy-plane.

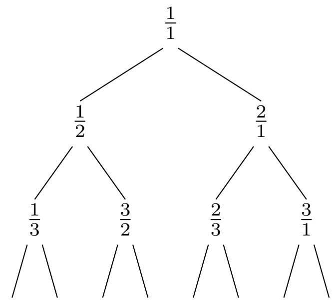
(1) Check whether the point $(2, \sqrt{2})$ is in the region A.

(2) Sketch the region A.

(3) Find the volume of the solid obtained by rotating the region A about the x-axis.

[4]

Given that a tree diagram whose starting node is $\dfrac{1}{1}$, and each node there is a fraction $\dfrac{p}{q}$ and next branches are $\dfrac{p}{p+q}$ at the left side and $\dfrac{p+q}{q}$ at the right side as below.



(1) All fractions at the nodes of this tree diagram are irreducible fractions. (We consider that $\dfrac{n}{1}$ is an irreducible fraction.)

(2) Show that all positive rational numbers are in this tree diagram.

(3) Each fractions in this tree diagram are different each other.

(4) Find the place of the fraction $\dfrac{19}{44}$ in this tree diagram. For example $\dfrac{3}{1}$ is at the third row and 4th column from the left side.

[5]

Given that two spheres:

$$S_1 : (x-1)^2 + (y-1)^2 + (z-1)^2 = 7$$

and

$$S_2 : (x-2)^2 + (y-3)^2 + (z-3)^2 = 1$$

Let C be the intersection of two spheres S_1 and S_2.

(1) In the group of spheres whose intersection with S_1 is C, find the equation of the sphere whose radius is the smallest.

(2) In the group of spheres whose intersection with S_1 is C, find the equation of the sphere whose radius is $\sqrt{3}$.