5 Sequeces and Series

5.1 Sequences

A sequence is a set of numbers occurring in order as:

$1, 3, 5, 7, \cdots$	the sequence of odd numbers
$1, 4, 9, 16, \cdots$	the sequence of perfect squares
$1, 1, 2, 3, 5, 8, 13, \cdots$	the Fibonacci sequence

A sequence are often represented by ordered letters like:

$$a_1,\ a_2,\ a_3,\ \cdots,\ a_n,\ \cdots$$

Such sequence is written as $\{a_n\}$.
a_1 is called the first term of the sequence, a_2 is the second term, ... and a_n is the n-th term.
For example the sequence of odd numbers,

$$a_1 = 1,\ a_2 = 3, \cdots,\ a_n = 2n - 1,\ \cdots$$

then the sequence of odd numbers is $\{2n - 1\}$

Example 1

Find the first four terms of the sequence whose n-th term is fiven by

$$a_n = \frac{1}{6}n(n+1)(2n+1)$$

Answer

Put $n = 1,\ 2,\ 3$ and 4 respectively into the expression for a_n,

$$a_1 = \frac{1}{6} \cdot 1 \cdot (1+1)(2 \cdot 1 + 1) = 1$$

$$a_2 = \frac{1}{6} \cdot 2 \cdot (2+1)(2 \cdot 2 + 1) = 5$$

$$a_3 = \frac{1}{6} \cdot 3 \cdot (3+1)(2 \cdot 3 + 1) = 14$$

$$a_4 = \frac{1}{6} \cdot 4 \cdot (4+1)(2 \cdot 4 + 1) = 30$$

Example 2

Find the n-th term of the sequence

$$-1,\ 2,\ -3,\ 4,\ \cdots$$

Answer

For changing from positive to negative alternatively, we use powers of -1:

$$(-1)^1 = -1,\ (-1)^2 = +1,\ (-1)^3 = -1,\ (-1)^4 = +1, \cdots$$

Then the n-th term of our sequence is

$$a_n = (-1)^n n$$

Recurence relation:

Sometimes the rule for defining a sequence is given in the form of a *recurrence relation.*
For example considering the Fibonacci sequence $\{a_n\}$:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \cdots$$

which is given as:

$$\begin{aligned} a_1 &= 1 & \\ a_2 &= 1 & \\ a_3 &= 2 &= a_1 + a_2 \\ a_4 &= 3 &= a_2 + a_3 \\ a_5 &= 5 &= a_3 + a_4 \\ a_6 &= 8 &= a_4 + a_5 \\ a_7 &= 13 &= a_4 + a_5 \end{aligned}$$

So we have its recurrence relation:

$$a_{n+2} = a_n + a_{n+1}$$

Then this Fibonacci sequence is defined by

$$a_1 = 1, \quad a_2 = 1 \quad \text{and} \quad [a_{n+2} = a_n + a_{n+1}$$

5.2 Summation

A summation is an addition of the terms of sequence:

$$1 + 3 + 5 + 7 + \cdots \qquad \text{is a summation of odd numbers}$$

For describing a summation we often use the sigma notation:

$$\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5$$

The k is a dummy variable, and you can change to another letter.

$$\sum_{k=1}^{5} a_k = \sum_{i=1}^{5} a_i = \sum_{r=1}^{5} a_r (= a_1 + a_2 + a_3 + a_4 + a_5)$$

The $k = 1$ at the foot of the sigma indicates the term with which the summation starts, and the 5 at the top indicates the term with which the summation finishes.
For example

$$\sum_{i=2}^{4} (2i - 1) = (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + (2 \cdot 4 - 1)$$

The summation starts with $i = 2$ and finishes with $i = 4$.

Example 3

(i) Express $1 + 4 + 9 + 18 + 25 + 36 + 49$ using sigma sign.

(ii) Evaluate $\displaystyle\sum_{k=0}^{5} k(k + 1)$

Answer

(i) The n-th term of the sequence $1,\ 4,\ 9,\ 18,\ 25,\ 36,\ 49$ is n^2.
And the summation starts with the first term and finishes with the 7th term. Then

$$1 + 4 + 9 + 18 + 25 + 36 + 49 = \sum_{k=1}^{7} k^2$$

(ii)

$$\begin{aligned}\sum_{k=0}^{5} k(k+1) &= 0\cdot(0+1)+1\cdot(1+1)+2\cdot(2+1)\\ &\quad +3\cdot(3+1)+4\cdot(4+1)+5\cdot(5+1)\\ &= 0+2+6+12+20+30\\ &= 70\end{aligned}$$

Linearity of sigma notation

(i) $\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n}(a_k + b_k)$

(ii) $\sum_{k=1}^{n} pa_k = p\sum_{k=1}^{n} a_k$

5.3 Arithmetic Progression

An arithmetic progression is a sequence such that the difference between consecutive terms is constant . For example

$$0,\ 5,\ 10,\ 15,\ \cdots$$

$$7,\ 4,\ 1,\ -2,\ \cdots$$

are arithmetic progressions.
The difference between consecutive terms is called a *common difference.*
Then for the first example, the common difference is 5
For the second example, the common difference is -3.

Let $\{a_n\}$ be an arithmetic progression, whose first term is $a_1 = a$ and the common difference is d. Then

$a_1 \quad = a$
$a_2 \quad = a + d$
$a_3 \quad = a_2 + d \quad = a + 2d$
$a_4 \quad = a_3 + d \quad = a + 3d$

Hence th n-th term of the arithmetic progression $\{a_n\}$ is

$$a_n = a + (n + 1)d$$

Example 4

(i) Write down the first five terms of the arithmetic progression with first term 7 and common difference -4. And write down the n-th term.

(ii) Given that an arithmetic progression $\{a_n\}$ by

$$a_1 = k,\ a_2 = \frac{2}{3}k,\ a_3 = \frac{1}{3}k,\ a_4 = 0,\ \cdots$$

a) Find the n-th term.

b) If the 20th term is equal to -16, find the value of k.

Answer

(1) Let $\{a_n\}$ be our sequence.

$a_1 \quad = 7$
$a_2 \quad = 7 - 4 = 3$
$a_3 \quad = 3 - 4 = -1$
$a_4 \quad = -1 - 4 = -5$
$a_5 \quad = -5 - 4 = -9$

And the n-th term is

$$a_n = 7 + (n - 1) \cdot (-4) = -4n + 11$$

(2) (i) The arithmetic progression has the first term k and common difference $-\frac{1}{3}k$. Then the n-th term is

$$\begin{aligned} a_n &= k + (n-1) \cdot (-\frac{1}{3}k) \\ &= -\frac{1}{3}kn + \frac{4}{3}k \end{aligned}$$

(ii) The 20th term is

$$a_{20} = -\frac{1}{3}k \cdot 20 + \frac{4}{3}k = -\frac{16}{3}k$$

$$-\frac{16}{3}k = -16, \qquad k = 3$$

Summation of an arithmetic progression

Now we consider about the sum of an arithmetic progression.

One day a maths teacher gave an exercise for his pupils. The exercise looks like this:

$$\text{Evaluate:} \quad 1 + 2 + 3 + 4 + \cdots + 100$$

A young Gauss (German mathematician: 1777- 1855) found the result immediately that it was 5050.

Why could he find this result immediately?
His method is like this:

$$(1 + 100) \times 100 \div 2 = 5050$$

This is just a method of summation of an arithmetic progression.

Let $S = 1 + 2 + 3 + 4 + \cdots + 100$
Then

$$\begin{array}{rllllll} & S & = 1 & +2 & +3 & +4 & + \cdots & +100 \\ +) & S & = 100 & +99 & +98 & +97 & + \cdots & +1 \\ \hline & 2S & = 101 & +101 & +101 & +101 & + \cdots & +101 \end{array}$$

$$2S = 101 \times 100 = 10100$$

$$S = 10100 \div 2 = 5050$$

Given that an arithmetic progression $\{a_n\}$.
Let $S_n = a_1 + a_2 + \cdots + a_n$, then

$$\begin{array}{rlllll} & S_n & = a_1 & +a_2 & +\cdots & +a_n \\ +) & S_n & = a_n & +a_{n-1} & +\cdots & +a_1 \\ \hline & 2S_n & = (a_1 + a_n) \times n & & & \end{array}$$

Hence

$$S_n = \frac{1}{2}n(a_1 + a_n)$$

Arithmetic progression

- The recurrence relation of an arithmetic progression is

$$a_{n+1} = a_n + d \quad \text{(where } d \text{ is a constant, called } \textit{common difference}\text{)}$$

- The n-th terms of an arithmetic progression $\{a_n\}$ is

$$a_n = a + (n-1)d$$

- The summation of an arithmetic progression $\{a_n\}$ from the first term to the n-th term is

$$S_n = \frac{1}{2}n(a_1 + a_n)$$

Example 5

(i) Let $\{a_n\}$ be an arithmetic progression, whose first term is 2 and common difference is 5.

  (i) Find the 15th term and the n-th term.

  (ii) Find the sum of the progression from the first term to the n-th term.

(ii) Given that an arithmetic progression $\{a_n\}$, whose 10th term is 67 and the 20th term is 137.

  (a) Find the first term and common difference.

  (b) Find the n-th term.

  (c) Find the sum of the progression from the first term to the 100th term.

Answer

(i) (i) The n-th term is

$$a_n = 2 + 5(n-1) = 5n - 3$$

Then the 15th term is

$$a_{15} = 5 \times 15 - 3 = 72$$

(ii) The sum of the progression is

$$S_n = \frac{1}{2}n(a_1 + a_n) = \frac{1}{2}n(2 + (5n - 3)) = \frac{n(5n-1)}{2}$$

(ii) (a) Let the first term of the progression $\{a_n\}$ be a and common difference be d. Then

$$a_n = a + (n-1)d$$

Since $a_{10} = 67$ and $a_{20} = 137$,

$$\begin{aligned} a_{10} &= a + 9d &= 67 \cdots (1) \\ a_{20} &= a + 19d &= 137 \cdots (2) \end{aligned}$$

(2)−(1)

$$10d = 70$$

$$d = 7, \qquad a = 4$$

(b)

$$a_n = 4 + 7(n-1) = 7n - 3$$

(c) The 100th term is $a_{100} = 7 \times 100 - 3 = 697$. Hence

$$S_{100} = \frac{1}{2} \times 100 \times (4 + 697) = 35050$$

Example 6

Let $\{a_n\}$ be an arithmetic progression, where $a_{19} = 230,\ a_{25} = 220$.
Let S_n be the sum of the arithmetic progression $\{a_n\}$ from the first term to the n-th term.
When S_n acieves to its maximum, find the value of n and the maximum of S_n.

Answer

Don't do that expressing first $S_n = \frac{1}{2}n(a_1 + a_n)$ as a quadratic function of n, then finding its maximum ...
It is possible to solve the problem like that, but calculation is very complicated.

The common difference is negative, then the terms of sequence become smaller and after sime term a_k, terms become negative.
It means

$$S_1 < S_2 < \cdots < S_k > S_{k+1} > S_{k+2} > \cdots$$

Then S_k must be the maximum.

Now we start our solution.

$$\begin{aligned} a_{19} &= a + 18d &= 230 \cdots (1) \\ a_{25} &= a + 24d &= 220 \cdots (2) \end{aligned}$$

$(2)-(1)$

$$6d=-10, \quad d=-\frac{5}{3} \qquad \text{and} \quad a=260$$

Hence the n-th term is

$$a_n=260-\frac{5}{3}(n-1)=-\frac{5}{3}n+\frac{785}{3}$$

If $a_n \leq 0$

$$-\frac{5}{3}n+\frac{785}{3} \leq 0, \quad n \geq \frac{785}{5}=157$$

That is $a_{156}>0,\ a_{157}=0$ and $a_{158}<0$.
Hence the S_n achieves the maximum, when $n=156$ or $n=157$
Since $a_{156}=-\frac{5}{3}\times 156+\frac{785}{3}=\frac{5}{3}$ or $a_{157}=0$

$$S_{156}=S_{157}=\frac{1}{2}\times 156\times(260+\frac{5}{3})=20410$$

5.4 Geometric Progression

A geometric progression is a sequence such that the ratio between consecutive terms is constant . For example

$$1,\ 3,\ 9,\ 27,\ \cdots$$

$$8,\ -4,\ 2,\ -1,\ \cdots$$

are geometric progressions.
The ratio between consecutive terms is called a *common ratio*.
Then for the first example, the common ratio is 3
For the second example, the common ratio is $-\frac{1}{2}$.

Let $\{a_n\}$ be a geometric progression, whose first term is $a_1 = a$ and the common ratio is r. Then

$a_1 \quad = a$
$a_2 \quad = ar$
$a_3 \quad = a_2 r \quad = ar^2$
$a_4 \quad = a_3 r \quad = ar^3$

Hence th n-th term of the geometric progression $\{a_n\}$ is

$$a_n = ar^{n-1}$$

Example 7

(i) Write down the first five terms of the geometric progression with first term 3 and common -2. And write down the n-th term.

(ii) Given that a geometric progression $\{a_n\}$, whose first term is 3 and common ration is 2. Find the 10th term and the n-th term.

Answer

(i)

$$a_1 = 3,\ a_2 = 3 \times (-2) = -6,\ a_3 = -6 \times (-2) = 12$$

$$a_4 = 12 \times (-2) = -24,\ a_5 = -24 \times (-2) = 48$$

And the n-th term is

$$a_n = 3 \cdot (-2)^{n-1}$$

(ii)

$$a_n = 3 \cdot 2^{n-1}$$

Hence

$$a_{10} = 3 \cdot 2^9 = 3 \times 512 = 1536$$

Summation of a geometric progression

Given that a geometric progression $\{a_n\}$, whose first term is a and common ration is r.
Let $S_n = a_1 + a_2 + \cdots + a_n$,

(i) If $r = 1$,
$a_n = a$ for all n, then

$$S_n = a + a + \cdots + a = na$$

(ii) If $r \neq 1$,

$$\begin{array}{rrlllllll} & S_n & = a & +ar & +ar^2 & +ar^3 & +\cdots & +ar^{n-1} & \\ -) & rS_n & = & +ar & +ar^2 & +ar^3 & +\cdots & +ar^{n-1} & +ar^n \\ \hline & (1-r)S_n & = a & & & & & & -ar^n \end{array}$$

Hence

$$S_n = \frac{a(1-r^n)}{1-r}$$

Geometric progression

- The recurrence relation of a geometric progression is

$$a_{n+1} = a_n r \quad \text{(where } r \text{ is a constant, called common ratio)}$$

- The n-th terms of a geometric progression $\{a_n\}$ is

$$a_n = ar^{n-1}$$

- The summation of a geometric progression $\{a_n\}$ from the first term to the n-th term is

$$S_n = \begin{cases} na & \text{if } r = 1 \\ \dfrac{a(1-r^n)}{1-r} & \text{if } r \neq 1 \end{cases}$$

Example 9

Find the n-th term of the following geometric progressions ,where a is the first term and r is common ratio. And find the sum of the geometric progression from the first term to the n-th term.

(i) $a = \frac{1}{4},\ r = -1$

(ii) $a = -3,\ r = \frac{1}{2}$

Answer

(i)

$$a_n = \frac{1}{4} \cdot (-1)^{n-1} = \frac{(-1)^n}{4}$$

$$S_n = \frac{\frac{1}{4}(1-(-1)^n)}{1-(-1)} = \frac{\frac{1}{4}(1-(-1)^n)}{2} = \frac{1-(-1)^n}{8}$$

(ii)

$$a_n = -3 \cdot (\frac{1}{2})^{n-1} = -3 \cdot \frac{1}{2^{n-1}} = -\frac{3}{2^{n-1}}$$

$$S_n = \frac{-3(1-(\frac{1}{2})^n)}{1-\frac{1}{2}} = \frac{-3(1-(\frac{1}{2})^n)}{\frac{1}{2}} = -6\left(1-(\frac{1}{2})^n\right)$$

Example 10

(i) Let $\{a_n\}$ be a geometric progression, whose fifth term is $\frac{3}{16}$ and the eighth term is $\frac{3}{128}$. Find the n-th term of this geometric progression.

(ii) Let S_n be expressed the sum of a geometric progression form the first term to the n-th term.
Given that $S_n = 24$ and $S_{2n} = 30$. Find the value of S_{3n}, the sum from the first term to the $3n$-th term.

Answer

(1) Let a be the first term and r be common ratio. Then

$$\begin{aligned} a_5 &= ar^4 &= \frac{3}{16} \cdots (1) \\ a_8 &= ar^7 &= \frac{3}{128} \cdots (2) \end{aligned}$$

(2) $\div$ (1)

$$\frac{ar^7}{ar^4} = \frac{3}{128} \div \frac{3}{16} = \frac{3}{128} \times \frac{16}{3}$$

$$r^3 = \frac{1}{8}$$

$$r = \frac{1}{2}, \qquad a = 3$$

Hence the n-th term is

$$a_n = 3 \cdot (\frac{1}{2})^{n-1} = \frac{3}{2^{n-1}}$$

(2) Let a be the first term and r be common ratio.

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} &= 24 \cdots (1) \\ S_{2n} &= \frac{a(1-r^{2n})}{1-r} &= 30 \cdots (2) \end{aligned}$$

(2) $\div$(1)

$$\frac{a(1-r^{2n})}{1-r} \div \frac{a(1-r^n)}{1-r} = \frac{30}{24}$$

$$\frac{a(1-r^{2n})}{1-r} \times \frac{1-r}{a(1-r^n)} = \frac{5}{4}$$

$$\frac{a(1+r^n)(1-r^n)}{1-r} \times \frac{1-r}{a(1-r^n)} = \frac{5}{4}$$

$$1 + r^n = \frac{5}{4}$$

$$r^n = \frac{1}{4}$$

From (1)

$$\frac{a(1-\frac{1}{4})}{1-r} = 24, \quad \frac{3}{4} \cdot \frac{a}{1-r} = 24$$

$$\frac{a}{1-r} = 32$$

Hence

$$S_{3n} = \frac{a(1-r^{3n})}{1-r} = \frac{a}{1-r} \cdot (1 - (\frac{1}{4})^3) = 32 \cdot \frac{63}{64} = \frac{63}{2}$$

5.5 Summation, Divers types of sequence

Summation of perfect squares, cubic numbers

(i) $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$

(ii) $\displaystyle\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

(iii) $\displaystyle\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{1}{2}n(n+1)\right)^2$

Proof

(i) The sum of arithmetic progression, whose first term is 1 and the n-th term is n, then

$$S_n = \frac{(1+n)n}{2} = \frac{1}{2}n(n+1)$$

(ii)

$$\begin{aligned}(k+1)^3 - k^3 &= (k^3 + 3k^2 + 3k + 1) - k^3 \\ &= 3k^2 + 3k + 1\end{aligned}$$

Then

$$\sum_{k=1}^{n}((k+1)^3 - k^3) = \sum_{k=1}^{n}(3k^2 + 3k + 1)$$

$$\sum_{k=1}^{n}((k+1)^3 - k^3) = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

where we can find

$\displaystyle\sum_{k=1}^{n}((k+1)^3 - k^3)$

$\displaystyle= \sum_{k=1}^{n}(-k^3 + (k+1)^3)$

$= (-1^3 + 2^3) + (-2^3 + 3^3) + (-3^3 + 4^3) + \cdots + (-n^3 + (n+1)^3)$

$= -1 + (n+1)^3$

$\displaystyle\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$

and

$\displaystyle\sum_{k=1}^{n} 1 = 1 + 1 + 1 + \cdots + 1 = n$

Hence

$$-1+(n+1)^3 = 3\sum_{k=1}^{n} k^2 + 3\cdot\frac{1}{2}n(n+1)+n$$

$$3\sum_{k=1}^{n} k^2 = ((n+1)^3-1) - \frac{3}{2}n(n+1) - n$$

$$3\sum_{k=1}^{n} k^2 = \frac{1}{2}(2n^3+3n^2+n) = \frac{1}{2}n(n+1)(2n+1)$$

Therefore

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

(iii)

$$\begin{aligned}(k+1)^4 - k^4 &= (k^4+4k^3+6k^2+4k+1) - k^4 \\ &= 4k^3+6k^2+4k+1\end{aligned}$$

Then

$$\sum_{k=1}^{n}((k+1)^4-k^4) = \sum_{k=1}^{n}(4k^3+6k^2+4k+1)$$

Hence

$$4\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n}((k+1)^4-k^4) - 6\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

$$\begin{aligned}4\sum_{k=1}^{n} k^3 &= ((n+1)^4-1) - n(n+1)(2n+1) - 2n(n+1) - n \\ &= (n^4+4n^3+6n^2+4n) - n(n+1)(2n+1) - 2n(n+1) - n \\ &= n(n^3+4n^2+6n+4) - (n+1)(2n+1) - 2(n+1) - 1) \\ &= n(n^3+2n^2+n) \\ &= n^2(n^2+2n+1) \\ &= n^2(n+1)^2\end{aligned}$$

Therefore

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2 = \left(\frac{1}{2}n(n+1)\right)$$

Example 11

Find the sum of the following sequences, from the first term to the n-th term.

(1) $1^2,\ 3^2,\ 5^2,\ 7^2,\ \cdots$

(2) $1\cdot 2\cdot 3,\ 3\cdot 4\cdot 5,\ 5\cdot 6\cdot 7,\ \cdots$

Answer

(1) The n-th term of the sequence is $(2n-1)^2$. Then the sum is

$$\begin{aligned}
\sum_{k=1}^{n}(2k-1)^2 &= \sum_{k=1}^{n}(4k^2-4k+1) \\
&= 4\sum_{k=1}^{n}k^2 - 4\sum_{k=1}^{n}k + \sum_{k=1}^{n}1 \\
&= 4\left(\frac{1}{6}n(n+1)(2n+1)\right) - 4\left(\frac{1}{2}n(n+1)\right) + n \\
&= \frac{1}{3}n\left(2(n+1)(2n+1) - 6(n+1) + 3\right) \\
&= \frac{1}{3}n(4n^2-1) \\
&= \frac{1}{3}n(2n+1)(2n-1)
\end{aligned}$$

(2) The n-th term of the sequence is $(2n-1)\cdot 2n \cdot (2n+1)$. Then the sum is

$$\begin{aligned}
&\sum_{k=1}^{n}(2k-1)2k(2k+1) \\
&= \sum_{k=1}^{n}(8k^3-2k) \\
&= 8\sum_{k=1}^{n}k^3 - 2\sum_{k=1}^{n}k \\
&= 8\frac{1}{4}n^2(n+1)^2 - 2\frac{1}{2}n(n+1) \\
&= 2n^2(n+1)^2 - n(n+1) \\
&= n(n+1)(2n(n+1)-1) \\
&= n(n+1)(2n^2+2n-1)
\end{aligned}$$

Example 12

Find the sum of the following sequences.

(i) $\displaystyle\sum_{k=1}^{n}\frac{1}{k(k+1)}$

(ii) $\displaystyle\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}$

Answer

(i) Since

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{n} - \frac{1}{n+1}) \\ &= 1 - \frac{1}{n+1} \\ &= \frac{n}{n+1}\end{aligned}$$

(ii) Since

$$\frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right)$$

$$\begin{aligned}&\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} \\ &= \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right) \\ &= \frac{1}{2}\left((\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3}) + (\frac{1}{2\cdot 3} - \frac{1}{3\cdot 4}) + (\frac{1}{3\cdot 4} - \frac{1}{4\cdot 5})\right. \\ &\left. + \cdots + (\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)})\right) \\ &= \frac{1}{2}\left(\frac{1}{1\cdot 2} - \frac{1}{(n+1)(n+2)}\right) \\ &= \frac{(n+1)(n+2) - 2}{4(n+1)(n+2)} \\ &= \frac{n(n+3)}{4(n+1)(n+2)}\end{aligned}$$

Sequence of differences

Let $\{a_n\}$ be a sequence. From this sequence we have a new sequence defined by

$$b_n = a_{n+1} - a_n$$

Such sequence $\{b_n\}$ is called the sequence of differences of the sequence $\{a_n\}$

For example a sequence of differences $\{b_n\}$ of an arithmetic progression $a_n = 2n - 1$ is a constant sequence

$$b_n = 2 \qquad \text{for all } \; n$$

Generally we can find the n-th term of the sequence $\{a_n\}$ using its sequence of differences $\{b_n\}$:

$$\begin{aligned} a_2 &= a_1 + b_1 \\ a_3 &= a_2 + b_2 &= a_1 = (b_1 + b_2) \\ a_4 &= a_3 + b_3 &= a_1 + (b_1 + b_2 + b_3) \end{aligned}$$

Then

$$a_n = a_1 + (b_1 + b_2 + b_3 + \cdots + b_{n-1}) \qquad (\text{for } n \geq 2)$$

Or we can write down as

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k \qquad (\text{for } n \geq 2)$$

Example 13

Find the n-th term of the following sequences.

(i) 2, 7, 18, 35, 58, 87, $\cdots$

(ii) 6, 8, 18, 42, 86, 156, 258, $\cdots$

Answer

(1) The sequence of differences $\{b_n\}$ is

$$5,\ 11,\ 17,\ 23,\ 29,\ \cdots$$

Then $\{b_n\}$ is an arithmetic progression with the first term is 5 and common difference is 6. Then

$$b_n = 5 + 6(n-1) = 6n - 1$$

Hence our sequence $\{a_n\}$ is

$$a_1 = 2$$

and when $n \geq 2$

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} b_k \\ &= 2 + \sum_{k=1}^{n-1} (6k - 1) \\ &= 2 + \left(6(\frac{1}{2}(n-1)n) - (n-1)\right) \\ &= 2 + (3n^2 - 4n + 1) \\ &= 3n^2 - 4n + 3 \end{aligned}$$

This expression includes the result $a_1 = 2$

Hence

$$a_n = 3n - 4n + 3$$

(2) The sequence of differences $\{b_n\}$ is

$$2,\ 10,\ 24,\ 44,\ 70,\ 102,\ \cdots$$

The sequence of differences $\{c_n\}$ of $\{b_n\}$ is

$$8,\ 14,\ 20,\ 26,\ 32,\ \cdots$$

The sequence $\{c_n\}$ is an arithmetic progression with the first term 8 and common differnce 6.

$$c_n = 8 + 6(n-1) = 6n + 2$$

Then the sequence $\{b_n\}$ is

$$b_1 = 2$$

And when $n \geq 2$

$$\begin{aligned} b_n &= b_1 + \sum_{k=1}^{n-1} c_k \\ &= 2 + \sum_{k=1}^{n-1} (6k+2) \\ &= 2 + \left(6(\frac{1}{2}(n-1)n) + 2(n-1)\right) \\ &= 2 + (3n^2 - n - 2) \\ &= 3n^2 - n \end{aligned}$$

This expression includes the result $b_1 = 2$
Then

$$b^n = 3n^2 - n$$

The sequence $\{a_n\}$ is

$$a_1 = 6$$

When $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} b_k \\ &= 6 + \sum_{k=1}^{n-1} (3k^2 - k) \\ &= 6 + \left(3(\frac{1}{6}(n-1)n(2(n-1)+1)) - \frac{1}{2}(n-1)n\right) \\ &= 6 + \frac{1}{2}n(n-1)((2n-1)-1) \\ &= 6 + n(n-1)^2 \\ &= n^3 - 2n^2 + n + 6 \end{aligned}$$

This expression includes the result $a_1 = 6$
Then

$$a_n = n^3 - 2n^2 + n + 6$$

Example 14

Given that a sequence $\{a_n\}$. Let $S_n = \sum_{k=1}^{n} a_k$.
Find the n-th term a_n of the sequences which satisfy the following condition.

(1) $S_n = n^2 - 2n$

(2) $S_n = 3n^2 + 1$

Answer

If $n \geq 2$,

$$\begin{array}{rrcllllll} & S_n & = & a_1 & +a_2 & +a_3 & +\cdots & +a_{n-1} & +a_n \\ -) & S_{n-1} & = & a_1 & +a_2 & +a_3 & +\cdots & +a_{n-1} & \\ \hline & S_n - S_{n-1} & = & & & & & & a_n \end{array}$$

Then

$$\begin{cases} a_1 & = S_1 \\ a_n & = S_n - S_{n-1} \qquad (n \geq 2) \end{cases}$$

(1) $a_1 = S_1 = 1^2 - 2 \cdot 1 = -1$
When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ &= (n^2 - 2n) - ((n-1)^2 - 2(n-1)) \\ &= (n^2 - 2n) - (n^2 - 4n + 3) \\ &= 2n - 3 \end{aligned}$$

This expression includes the result $a_1 = -1$
Hence

$$a_n = 2n - 3$$

(2) $a_1 = S_1 = 3 \cdot 1^2 + 1 = 4$
When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ &= (3n^2 + 1) - (3(n-1)^2 + 1) \\ &= (3n^2 + 1) - (3n^2 - 6n + 4) \\ &= 6n - 3 \end{aligned}$$

When you put $n = 1$ in this expression $a_1 = 3$, not 4.
Hence

$$a_n = \begin{cases} 4 & (\text{when } n = 1) \\ 6n - 3 & (\text{when } n \geq 2) \end{cases}$$

5.6 Recurrence Relation

In this section we shall see how to find the$n$-th term of a sequence from its recurrence relations.

- $\boldsymbol{a_{n+1} = a_n + d}$

  The sequence $\{a_n\}$ is an arithmetic progression with common difference d.
  Then
  $$a_n = a + (n-1)d$$

- $\boldsymbol{a_{n+1} = ra_n}$

  The sequence $\{a_n\}$ is a geometric progression with common ratio r.
  Then
  $$a_n = ar^{n-1}$$

- $\boldsymbol{a_{n+1} = a_n + b_n}$

  The sequence $\{b_n\}$ is the sequence of differences of $\{a_n\}$.
  Then
  $$a_n = a_1 + \sum_{k=1}^{n-1} b_k \qquad \text{(for } n \geq 2\text{)}$$

For these three type we can find direct the n-th term of the sequence.

Example 15

Find the n-th term of the following sequences, which are given in the recurrence relation.

(i) $a_1 = 3, \qquad a_{n+1} = a_n - 5$

(ii) $a_1 = -2, \qquad a_{n+1} = \frac{1}{2}a_n$

(iii) $a_1 = 1, \qquad a_{n+1} = a_n + 2n - 1$

Answer

(i) Since $a_{n+1} = a_n - 5$, the sequence $\{a_n\}$ is an arithmetic progression with common difference -5.
Hence
$$a_n = 3 + (-5)(n-1) = -5n + 8$$

(ii) Since $a_{n+1} = \frac{1}{2}a_n$, the sequence $\{a_n\}$ is a geometric progression with common ratio $\frac{1}{2}$.
Hence

$$a_n = (-2) \cdot (\frac{1}{2})^{n-1} = -\frac{2}{2^{n-1}} = -\frac{1}{2^{n-2}}$$

(iii) Since $a_{n+1} = a_n + 2n - 1$, the sequence $\{2n-1\}$ is the sequence of differnces of $\{a_n\}$. Hence when $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1}(2k-1) \\ &= 1 + (2 \cdot \frac{1}{2}(n-1)n - (n-1)) \\ &= n^2 - 2n + 2 \end{aligned}$$

This expression includes the value $a_1 = 1$.
Hence

$$a_n = n^2 - 2n + 2$$

- $\boldsymbol{a_{n+1} = pa_n + q}$

For recurrence relation $a_{n+1} = pa_n + q$,
we solve the equation $\alpha = p\alpha + q$ for finding a value of α.
Then we have the relation

$$a_{n+1} - \alpha = p(a_n - \alpha)$$

It means the sequence $\{a_n - \alpha\}$ is a geometric progression.

Example 16

Find the n-th term of the sequence,which is given in the recurrence relation:

$$a_1 = 2, \qquad a_{n+1} = 2a_n + 1$$

Answer

Solve the equation $\alpha = 2\alpha + 1, \quad \alpha = -1$
Then the recurrence relation is written as

$$a_{n+1} + 1 = 2(a_n + 1)$$

Then $\{a_n + 1\}$ is a geometric progression with common ratio 2.

$$a_n + 1 = (a_1 + 1) \cdot 2^{n-1}$$

$$a_n + 1 = (2 + 1) \cdot 2^{n-1}$$

$$a_n = 3 \cdot 2^{n-1} - 1$$

- $\boldsymbol{a_{n+1} = pa_n + q^n}$

Dividing by q^{n+1},

$$\frac{a_{n+1}}{q^{n+1}} = \frac{pa_n}{q^{n+1}} + \frac{q^n}{q^{n+1}}$$

$$\frac{a_{n+1}}{q^{n+1}} = \frac{p}{q}(\frac{a_n}{q^n}) + \frac{1}{q}$$

Put $b_n = \dfrac{a_n}{q^n}$, we have

$$b_{n+1} = \frac{p}{q}b_n + \frac{1}{q}$$

Then it becomes the former type of recurrence relation.

Example 17

Find the n-th term of the sequence,which is given in the recurrence relation:

$$a_1 = 2, \qquad a_{n+1} = 2a_n + 3^{n-1}$$

Answer

$a_{n+1} = 2a_n + 3^{n-1}$
Dividing by 3^{n+1},

$$\frac{a_{n+1}}{3^{n+1}} = \frac{2a_n}{3^{n+1}} + \frac{3^{n-1}}{3^{n+1}}$$

$$\frac{a_{n+1}}{3^{n+1}} = \frac{2}{3}(\frac{a_n}{3^n}) + \frac{1}{9}$$

Solve the equtions $\alpha = \dfrac{2}{3}\alpha + \dfrac{1}{9}$, $\quad \alpha = \dfrac{1}{3}$
Then the relation is expressed as

$$\frac{a_{n+1}}{3^{n+1}} - \frac{1}{3} = \frac{2}{3}\left(\frac{a_n}{3^n} - \frac{1}{3}\right)$$

$\{\dfrac{a_n}{3^n} - \dfrac{1}{3}\}$ is a geometric progression with common ratio $\dfrac{2}{3}$.
Then

$$\frac{a_n}{3^n} - \frac{1}{3} = \left(\frac{a_1}{3^1} - \frac{1}{3}\right)(\frac{2}{3})^{n-1}$$

$$\frac{a_n}{3^n} = \frac{2^{n-1}(2-1)}{3^n} + \frac{1}{3}$$

$$a_n = 2^{n-1} + 3^{n-1}$$

- $\boldsymbol{a_{n+1} = pa_n + (cn + d)}$

Substruction.

$$\begin{aligned} a_{n+2} - a_{n+1} &= (pa_{n+1} + (c(n+1) + d)) - (pa_n + (cn + d)) \\ &= p(a_{n+1} - a_n) + c \end{aligned}$$

Put $b_n = a_{n+1} - a_n$, then

$$b_{n+1} = pb_n + c$$

It is the former recurrence relation, and we continue to find the n-th term.

Example 18

Find the n-th term of the sequence,which is given in the recurrence relation:

$$a_1 = 2, \qquad a_{n+1} = 2a_n + 3n - 1$$

Answer

$$\begin{aligned} a_{n+2} - a_{n+1} &= (2a_{n+1} + (3(n+1) - 1)) - (2a_n + (3n - 1)) \\ &= 2(a_{n+1} - a_n) + 3 \end{aligned}$$

Put $b_n = a_{n+1} - a_n$, we have

$$b_{n+1} = 2b_n + 3$$

Solve the equation $\alpha = 2\alpha + 3, \qquad \alpha = -3.$
Then

$$b_{n+1} + 3 = 2(b_n + 3)$$

$\{b_n + 3\}$ is a geometric progression with common ratio 2.

$$b_n + 3 = b_1 \cdot 2^{n-1}$$

$$b_n = b_1 \cdot 2^{n-1} - 3$$

$$a_{n+1} - a_n = (a_2 - a_1) \cdot 2^{n-1} - 3$$

$a_2 = 2a_1 + 3 \cdot 1 - 1 = 4 + 3 - 1 = 6$, then

$$\begin{aligned} a_n &= 2 + \sum_{k=1}^{n-1} (6 \cdot 2^{n-1} - 3) \qquad (n \geq 2) \\ &= 2 + \frac{6(1 - 2^{n-1})}{1 - 2} - 3(n - 1) \\ &= 2 - 6(1 - 2^{n-1}) - 3n + 3 \\ &= 6 \cdot 2^{n-1} - 3n - 1 \\ &= 3 \cdot 2^n - 3n - 1 \end{aligned}$$

This expression includes the value $a_1 = 2$. Hence

$$a_n = 3 \cdot 2^n - 3n - 1$$

- $\boldsymbol{a_{n+1} = \dfrac{a_n}{pa_n + q}}$

  If $a_n \neq 0$ for all n, take the inverse:

$$\frac{1}{a_{n+1}} = \frac{pa_n + q}{a_n} = p + q\frac{1}{a_n}$$

  Put $b_n = \dfrac{1}{a_n}$, we have

$$b_{n+1} = p + qb_n$$

  which is in the former recurrence relation.

Example 19

Find the n-th term of the sequence,which is given in the recurrence relation:

$$a_1 = 2, \qquad a_{n+1} = \frac{a_n}{2a_n + 3}$$

Answer

$a_{n+1} = \dfrac{a_n}{2a_n + 3}$
$a_1 > 0$.
If $a_n > 0$ then $a_1 > 0$ and $2a_n + 3 > 0$, hence $a_{n+1} > 0$. By induction we prove that $a_n > 0$ for all n. hence $a_n \neq 0$.

Take the inverse of the recurrence relation:

$$\frac{1}{a_{n+1}} = \frac{2a_n + 3}{a_n} = 2 + 3\frac{1}{a_n}$$

Put $b_n = \dfrac{1}{a_n}$,

$$b_{n+1} = 3b_n + 2$$

Solve the equation $\alpha = 3\alpha + 2, \qquad \alpha = -1$. Then

$$b_{n+1} + 1 = 3(b_n + 1)$$

$\{b_n+1\}$ is a geometric progression with common ratio 3.
And $b_1 = \frac{1}{a_1} = \frac{1}{2}$.
Therefore

$$\begin{aligned} b_n + 1 &= (b_1 + 1) \cdot 3^{n-1} \\ &= (\frac{1}{2} + 1) \cdot 3^{n-1} \\ &= \frac{3^n}{2} \end{aligned}$$

Hence

$$b_n = \frac{3^n}{2} - 1 = \frac{3^n - 2}{2}$$

$$\frac{1}{a_n} = \frac{3^n - 2}{2}$$

Hence

$$a_n = \frac{2}{3^n - 2}$$

- $\boldsymbol{a_{n+2} = pa_{n+1} + qa_n}$

First we shall see how to transform the recurrence relation to the relation of geometric progression: $A_{n+1} = RA_n$

Example 20

When $a_{n+2} = pa_{n+1} + qa_n$ is transformed in the form

$$a_{n+2} - \alpha a_{n+1} = \beta(a_{n+1} - \alpha a_n)$$

Find the values of α and β.

Answer

From the expression $a_{n+2} - \alpha a_{n+1} = \beta(a_{n+1} - \alpha a_n)$,
we have

$$a_{n+2} = (\alpha + \beta)a_{n+1} - \alpha\beta a_n$$

Comparing the coefficients with the expression $a_{n+2} = pa_{n+1} + qa_n$,

$$\alpha + \beta = p, \quad \text{and} \quad \alpha\beta = -q$$

Then α and β are the roots of quadratic equation:

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

$$x^2 - px - q = 0$$

Or

$$x^2 = px + q$$

This quadrtic equation is called the *auxiliary equation*.

When we solve the auxiliary equation $x^2 = px + q$, we can find the values of α and β.

And our expression is transformed to $a_{n+2} - \alpha a_{n+1} = \beta(a_{n+1} - \alpha a_n)$,
The sequence $\{a_{n+1} - \alpha a_n\}$ is a geometric progression with common ratio β.

Example 21

Find the n-th term of the sequence,which is given in the recurrence relation:

$$a_1 = 1, \quad a_2 = 2, \quad a_{n+2} = 2a_{n+1} + 3a_n$$

Answer

Solve a quadratic equation $x^2 = 2x + 3$.

$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$
$$x = 3 \qquad \text{or} \qquad x = -1$$

Then $a_{n+2} = 2a_{n+1} + 3a_n$ is transformed in the form

$$a_{n+2} - 3a_{n+1} = -(a_{n+1} - 3a_n) \cdots (1)$$

Or

$$a_{n+2} + a_{n+1} = 3(a_{n+1} + a_n) \cdots (2)$$

From (1), we consider that $\{a_{n+1} - 3a_n\}$ is a geometric progression with common ratio -1.
Then

$$\begin{aligned} a_{n+1} - 3a_n &= (a_2 - 3a_1)(-1)^{n-1} \\ &= (2 - 3) \cdot 1)(-1)^{n-1} \\ &= (-1)^n \cdots (3) \end{aligned}$$

From (2), we consider that $\{a_{n+1} + a_n\}$ is a geometric progression with common ratio 3.
Then

$$\begin{aligned} a_{n+1} + a_n &= (a_2 + a_1)3^{n-1} \\ &= (2 + 1) \cdot 3^{n-1} \\ &= 3^n \cdots (4) \end{aligned}$$

(4) $-$(3)

$$\begin{array}{rrl} & a_{n+1} + a_n & = 3^n \\ -) & a_{n+1} - 3a_n & = (-1)^n \\ \hline & 4a_n & = 3^n - (-1)^n \end{array}$$

Hence

$$a_n = \frac{3^n - (-1)^n}{4}$$

Example 22

Give that a sequence whose recurrence relation is

$$a_1 = 0, \quad a_{n+1} = \frac{1 + a_n}{3 - a_n} \quad (n \geq 2)$$

(i) Find the values of a_2, a_3 and a_4

(ii) From the result of (i), assume that the expression of the n-th term of a_n.

(iii) Prove the assumption (ii) by induction.

Answer

(i)

$$a_2 = \frac{1+a_1}{3+a_1} = \frac{1+0}{3-0} = \frac{1}{3}$$

$$a_3 = \frac{1+a_2}{3+a_2} = \frac{1+\frac{1}{3}}{3-\frac{1}{3}} = \frac{3+1}{9-1} = \frac{1}{2} = \frac{2}{4}$$

$$a_4 = \frac{1+a_3}{3+a_3} = \frac{1+\frac{1}{2}}{3-\frac{1}{2}} = \frac{2+1}{6-1} = \frac{3}{5}$$

(ii) From (1) we assume that

$$a_n = \frac{n-1}{n+1} \cdots (*)$$

(iii) When $n = 1$, (*) is true for $a_1 = 0$.
Assume that (*) is true for n, that is $a_n = \dfrac{n-1}{n+1}$
Then

$$\begin{aligned} a_{n+1} &= \frac{1+a_n}{3-a_n} \\ &= \frac{1+\frac{n-1}{n+1}}{3-\frac{n-1}{n+1}} \\ &= \frac{(n+1)+(n-1)}{3(n+1)-(n-1)} \\ &= \frac{2n}{2n+4} \\ &= \frac{n}{n+2} \end{aligned}$$

Then (*) is true for $n+1$.
Hence, for all n, (*) is true.

5.7 Limit of sequences, Series

5.7.1 Limit of sequences

We shall examine what happens to a sequence as n gets very large.
There are four types of behaviour that we shall see. These are

(i) sequences that tend to a real number

(ii) sequences that tend to infinity

(iii) sequences that tend to minus infinity

(iv) sequences that does not tend to any number nor infinities

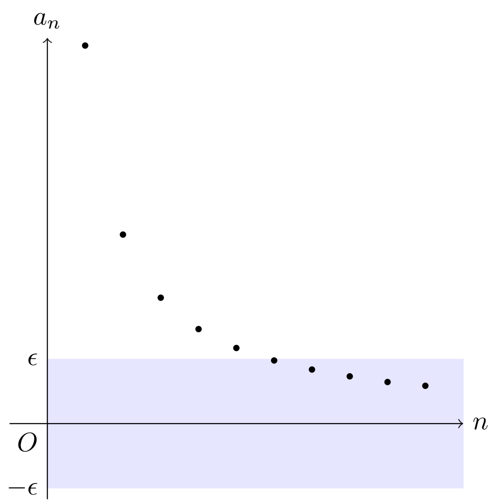
For example:

(i) The sequence

$$a_n : 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$$

tends to 0, as n gets very large.
We describe this as

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$



For any positive number $\epsilon > 0$, there exists an integer N, such that

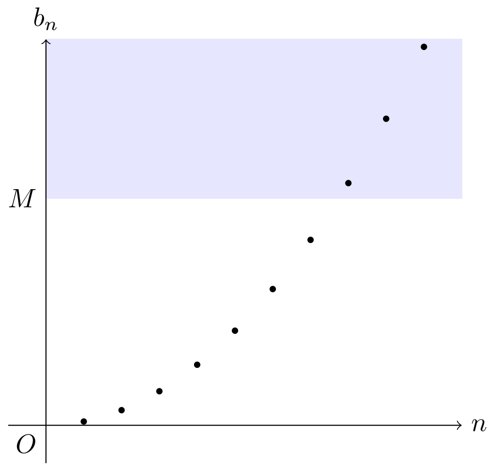
$$|a_n - 0| < \epsilon \quad \text{for all} \quad n \geq N$$

(ii) The sequence

$$b_n:\ 1^2,\ 2^2,\ 3^2,\ \cdots,\ n^2, \ldots$$

tends to infinity, as n gets very large:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} n^2 = \infty$$



For any positive number $M > 0$, there exists an integer N, such that

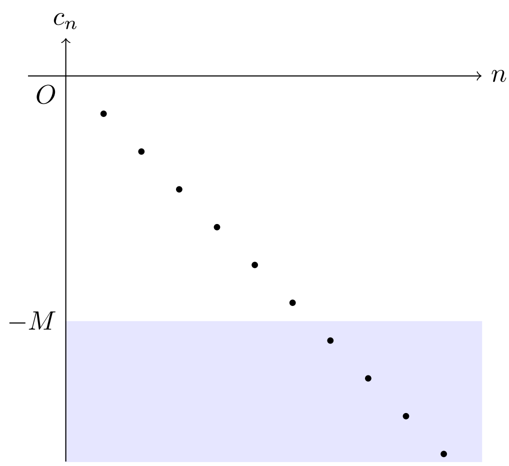
$$b_n > M \quad \text{for all} \quad n \geq N$$

(iii) The sequence

$$c_n:\ -2,\ -4,\ -6,\ \cdots,\ -2n,\ \cdots$$

tends to minus infinity, as n gets very large:

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} (-2n) = -\infty$$



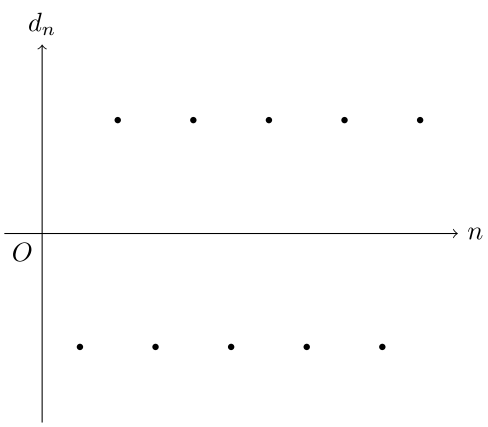
For any positive number $M > 0$, there exists an integer N, such that

$$c_n < -M \quad \text{for all} \quad n \geq N$$

(iv) The sequence

$$d_n : \ -3, \ 3, \ -3, \ 3, \ \cdots, \ (-1)^n \cdot 3, \ \cdots$$

does not tend to any number, as n gets very large.
It oscillates between -3 and 3 eternally, and such a sequence has no limits.



The sequarnce (i) is said to be *convergent*.
The sequence (ii), (iii) and (iv) is said to be *divergent*.

Convergence of sequence

A sequence $\{a_n\}$ is said to be convergent to a, if and only if

For any number $\epsilon > 0$, there exists a positive integer $N > 0$, such that

$$|a_n - a| < \epsilon \quad \text{for all} \quad n > N$$

And we shall describe this as:

$$\lim_{n\to\infty} a_n = a$$

Example 23

Find the limits:

(i) $\displaystyle\lim_{n\to\infty} 2^n$

(ii) $\displaystyle\lim_{n\to\infty} (100 - n)$

(iii) $\displaystyle\lim_{n\to\infty} \frac{n}{n+1}$

(iv) $\displaystyle\lim_{n\to\infty} (-3)^n$

Answer

(i) $\displaystyle\lim_{n\to\infty} 2^n = \infty$

(ii) $\displaystyle\lim_{n\to\infty} (100 - n) = -\infty$

(iii) $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$

(iv) The sequence $\{(-3)^n\}$ is

$$-3,\ 9,\ -27,\ 81,\ -243,\ \cdots$$

Then $\displaystyle\lim_{n\to\infty} (-3)^n$ has no limit.

The example (iii) is a type that

$$\frac{\text{(nominator)} \to \infty}{\text{(denominator)} \to \infty}$$

For finding its limit you will divide the top and the bottom of fraction by the largest index of n.

Example 24

Find the limit of the following expressions.

(i) $\lim_{n\to\infty} \frac{2n^3+4n^2-5}{3n^3-n+2}$

(ii) $\lim_{n\to\infty} \frac{(n-1)(-n^2+4)}{n^2-n+7}$

(iii) $\lim_{n\to\infty} (\log_2(4n^2-1) - \log_2(n^2+5n))$

Answer

(i) $\lim_{n\to\infty} \frac{2n^3+4n^2-5}{3n^3-n+2}$

We will divide the top and the bottom of fraction by n^3

$$\lim_{n\to\infty} \frac{2n^3+4n^2-5}{3n^3-n+2} = \lim_{n\to\infty} \frac{2+\frac{4}{n}-\frac{5}{n^2}}{3-\frac{1}{n}+\frac{2}{n^2}}$$

Since $\lim_{n\to\infty} \frac{a}{n} = 0, \quad \lim_{n\to\infty} \frac{b}{n^2} = 0 \quad (a,\ b \text{ are constants})$

$$\lim_{n\to\infty} \frac{2n^3+4n^2-5}{3n^3-n+2} = \lim_{n\to\infty} \frac{2+\frac{4}{n}-\frac{5}{n^2}}{3-\frac{1}{n}+\frac{2}{n^2}} = \frac{2}{3}$$

(ii) $\lim_{n\to\infty} \frac{(n-1)(-n^2+4)}{n^2-n+7}$

We will divide the top and the bottom of fraction by n^2

$$\begin{aligned}\lim_{n\to\infty} \frac{(n-1)(-n^2+4)}{n^2-n+7} &= \lim_{n\to\infty} \frac{-n^3+n^2+4n-4}{n^2-n+7} \\ &= \lim_{n\to\infty} \frac{-n+1+\frac{4}{n}-\frac{4}{n^2}}{1-\frac{1}{n}+\frac{7}{n^2}} \\ &= -\infty\end{aligned}$$

(iii) $\lim_{n\to\infty}(\log_2(4n^2-1)-\log_2(n^2+5n))$
$= \lim_{n\to\infty} \log_2 \frac{4n^2-1}{n^2+5n}$
$= \lim_{n\to\infty} \log_2 \frac{4-\frac{1}{n^2}}{1+\frac{5}{n}}$
$= \log_2 4$
$= 2$

Example 25

Find the limit of the following expressions.

(i) $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n-1})$

(ii) $\lim_{n\to\infty}(\sqrt{4n^2-5n+1}-2n)$

(iii) $\lim_{n\to\infty} \frac{\sqrt{n+3}-\sqrt{n+2}}{\sqrt{n+1}-\sqrt{n}}$

Answer

If a sequence looks like $A_n - B_n$ and each A_n and B_n tends to infinity, we cannot find the limit $\lim_{n\to\infty}(A_n - B_n) = \infty - \infty$?

(i) $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n-1})$
Rationatise the nominator.

$\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n-1})$
$= \lim_{n\to\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{1}$
$= \lim_{n\to\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{1} \cdot \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}}$
$= \lim_{n\to\infty} \frac{(n+1)-(n-1)}{\sqrt{n+1}+\sqrt{n-1}}$
$= \lim_{n\to\infty} \frac{2}{\sqrt{n+1}+\sqrt{n-1}}$
$= 0$

(ii) $\lim_{n\to\infty}(\sqrt{4n^2-5n+1}-2n)$

$= \lim_{n\to\infty}(\sqrt{4n^2-5n+1}-2n)\cdot\frac{\sqrt{4n^2-5n+1}+2n}{\sqrt{4n^2-5n+1}+2n}$

$= \lim_{n\to\infty}\frac{(4n^2-5n+1)-4n^2}{\sqrt{4n^2-5n+1}+2n}$

$= \lim_{n\to\infty}\frac{-5n+1}{\sqrt{4n^2-5n+1}+2n}$

We will divide the top and the bottom of fraction by n $(=\sqrt{n^2})$

$= \lim_{n\to\infty}\frac{-5+\frac{1}{n}}{\sqrt{4-\frac{5}{n}+\frac{1}{n^2}}+2}$

$= \frac{-5}{\sqrt{4}+2}$

$= -\frac{5}{4}$

(iii) We need rationailize both the top and the bottom of the fraction.

$\lim_{n\to\infty}\frac{\sqrt{n+3}-\sqrt{n+2}}{\sqrt{n+1}-\sqrt{n}}$

$= \lim_{n\to\infty}\frac{\sqrt{n+3}-\sqrt{n+2}}{\sqrt{n+1}-\sqrt{n}}\cdot\frac{(\sqrt{n+3}+\sqrt{n+2})(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+3}+\sqrt{n+2})(\sqrt{n+1}+\sqrt{n})}$

$= \lim_{n\to\infty}\frac{((n+3)-(n+2))(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+3}+\sqrt{n+2})((n+1)-n)}$

$= \lim_{n\to\infty}\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+3}+\sqrt{n+2}}$

We will divide the top and the bottom of fraction by $\sqrt{n}$

$= \lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}}+1}{\sqrt{1+\frac{3}{n}}+\sqrt{1+\frac{2}{n}}}$

$= \frac{\sqrt{1}+1}{\sqrt{1}+\sqrt{1}}$

$= 1$

5.7.2 Limit of Geometric Progressions

We shall see a behaviour about the sequence

$$a_n = r^n \qquad \text{where } r \text{ is a real number}$$

(i) When $r = 2$

$$a_n: \ 2, \ 4, \ 8, \ 16, \ 32, \ \cdots$$

tends to infinity.

$$\lim_{n\to\infty} 2^n = \infty$$

(ii) When $r = 1$

$$a_n: \ 1, \ 1, \ 1, \ 1, \ 1, \ \cdots$$

tends to 1.

$$\lim_{n\to\infty} 1^n = 1$$

(iii) When $r = \frac{1}{2}$

$$a_n: \ \frac{1}{2}, \ \frac{1}{4}, \ \frac{1}{8}, \ \frac{1}{16}, \ \frac{1}{32}, \ \cdots$$

tends to 0.

$$\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$$

(iv) When $r = -\frac{1}{2}$

$$a_n: \ -\frac{1}{2}, \ \frac{1}{4}, \ -\frac{1}{8}, \ \frac{1}{16}, \ -\frac{1}{32}, \ \cdots$$

tends to 0.

$$\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$$

(v) When $r = -1$

$$a_n: \ -1, \ 1, \ -1, \ 1, \ -1, \ \cdots$$

has no limit.

(vi) When $r = -2$

$$a_n: \ -2, \ 4, \ -8, \ 16, \ -32, \ \cdots$$

has no limit.

Limit of Geometric Progressions

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & (r > 1) \\ 0 & (-1 < r \leq 1) \\ \text{no limit} & (r \leq -1) \end{cases}$$

Example 26

Find the limit.

(i) $\lim_{n\to\infty} \dfrac{(-3)^n}{4^n - 3}$

(ii) $\lim_{n\to\infty} \dfrac{(-5)^n}{4^n - 3}$

(iii) $\lim_{n\to\infty} (2^{4n} - 5^{2n})$

Answer

(i) $\lim_{n\to\infty} \dfrac{(-3)^n}{4^n - 3} = \lim_{n\to\infty} \dfrac{(-\frac{3}{4})^n}{1 - 3 \cdot (\frac{1}{4})^n} = \dfrac{0}{1} = 0$

(ii) $\lim_{n\to\infty} \dfrac{(-5)^n}{4^n - 3} = \lim_{n\to\infty} \dfrac{(-\frac{5}{4})^n}{1 - 3 \cdot (\frac{1}{4})^n}$

The denominator tends to 1, but the nominator has no limit.
Then our equaence has no limit.

(iii) $\lim_{n\to\infty} (2^{4n} - 5^{2n}) = \lim_{n\to\infty} (16^n - 25^n) = \lim_{n\to\infty} 25^n((\dfrac{16}{25})^n - 1) = -\infty$

Example 27

Find the range of x, for which the geometric progression $\left\{ \left(\dfrac{3-x}{4} \right)^n \right\}$ converges.

Answer

For $\left\{ \left(\dfrac{3-x}{4} \right)^n \right\}$ converges, the common ratio $\dfrac{3-x}{x}$ must be:

$$-1 < \frac{3-x}{4} \leq 1$$

$$-4 < 3 - x \leq 4$$

$$-7 < -x \leq 1$$

Hence

$$-1 \leq x < 7$$

5.7.3 Series

Let $\{a_n\}$ be a sequence. We can form a new sequence as

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\cdots \\ s_n &= a_1 + a_2 + \cdots + a_n \\ &\cdots \end{aligned}$$

Such sequence $\{s_n\}$ is called *series* of the sequence $\{a_n\}$.
The n-th term of the sequence $\{s_n\}$ os called a partial sum of the series.

If $\{s_n\}$ converges to a number s, then s is called the sum of a series $\{a_n\}$, and we describe it as

$$\sum_{n=1}^{\infty} a_n = s$$

The sum of series $\sum_{n=1}^{\infty} a_n$ can be calculated as

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

that is, first you evaluate the partial sum of the series, next take its limit.

Example 28

Check whether the following series converge or diverge. For convergent series, find their sum.

(i) $(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots$

(ii) $\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots$

(iii) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$

Answer

(i) The partial sum s_n

$$\begin{aligned} s_n &= (1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\cdots+(\frac{1}{n}-\frac{1}{n+1}) \\ &= 1-\frac{1}{n+1} \end{aligned}$$

Then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}(1-\frac{1}{n+1}) = 1$$

Hence the series converges and its sum is

$$\sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n+1}) = 1$$

(ii) The partial sum s_n is

$$\begin{aligned} s_n &= \frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{n}+\sqrt{n+1}} \\ &= (-1+\sqrt{2})+(-\sqrt{2}+\sqrt{3})+(-\sqrt{3}+\sqrt{4}) \\ &\qquad +\cdots+(-\sqrt{n}+\sqrt{n+1}) \\ &= -1+\sqrt{n+1} \end{aligned}$$

Then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}(-1+\sqrt{n+1}) = \infty$$

Hence the series diverges.

(iii) The partial sum s_n is

$$\begin{aligned} s_n &= \frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)} \\ &= \frac{1}{2}\left((\frac{1}{1}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{5})+(\frac{1}{5}-\frac{1}{7})+\cdots+(\frac{1}{2n-1}-\frac{1}{2n+1})\right) \\ &= \frac{1}{2}\left(1-\frac{1}{2n+1}\right) \end{aligned}$$

Then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{2n+1}\right) = \frac{1}{2}$$

Hence the series converges and its sum is

$$\sum_{n=1}^{\infty}(\frac{1}{2n-1}-\frac{1}{2n+1}) = \frac{1}{2}$$

5.7.4 Geometric series

The geometric series is a series of a geometric progression.
The summation of geometric progression is

$$S_n = \begin{cases} na & \text{if } r = 1 \\ \dfrac{a(1-r^n)}{1-r} & \text{if } r \neq 1 \end{cases}$$

Then

(i) If $r = 1$
The partial sum is $s_n = na$.
$\lim_{n\to\infty} s_n = \infty \quad (a > 0)$ or $\lim_{n\to\infty} s_n = -\infty \quad (a < 0)$
Then the series diverges.

(ii) If $r \neq 1$
The partial sum is $s_n = \dfrac{a(1-r^n)}{1-r}$
It converges if and only if $-1 < r < 1$
Ans at this time the sum of series is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \dfrac{a}{1-r} & (-1 < r < 1) \\ \text{divergent} & (r \leq -1,\ 1 \leq r) \end{cases}$$

Example 29

Check whether the following geometric series converges or not. If it converges, find its sum.

(i) $6 + 2 + \dfrac{2}{3} + \dfrac{2}{9} + \cdots$

(ii) $5 - 5\sqrt{3} + 15 - 15\sqrt{3} + \cdots$

Answer

(i) The common ratio is $r = \dfrac{1}{3}$, which satisfies $-1 < r < 1$.
Hence the series converges and its sum is

$$\sum_{n=1}^{\infty} 6 \cdot (\frac{1}{3})^{n-1} = \frac{6}{1-\frac{1}{3}} = 9$$

(ii) The common ratio is $-\sqrt{3}$ (< -1), then the series diverges.

Example 30

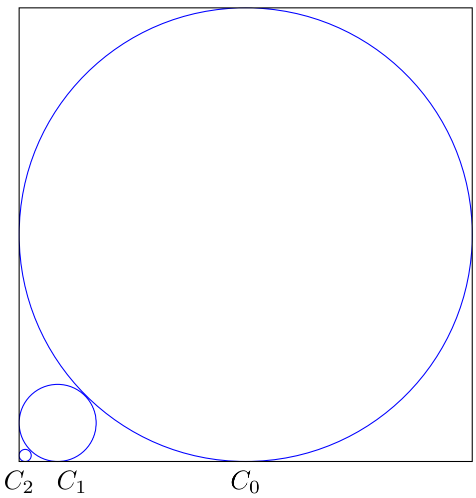
Let C_0 be an inscribed circle of the square R, whose length of the side is $2a$.
Let C_1 be a circle, which touches to the two sides of the square R and touches to the circle C_0.
Let C_2 be a circle, which touches to the two sides of the square R and touches to the circle C_1.
Continue this, and let C_{n+1} be a circle which touches to the two sides of the square R and touches to the circle C_n.

(i) Express the radius of the circle C_n in terms of n.

(ii) Find the sum of the series:

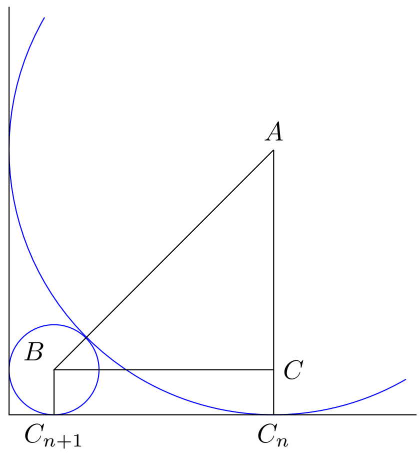
$$C_0 + C_1 + C_2 + \cdots$$

where C_n is represent an area of a circle C_n.



Answer

(i)



Let r_n be a radius of the circle C_n.
Then

$$AB = r_n + r_{n+1}$$
$$AC = r_n - r_{n+1}$$

Since $\angle ABC = 45°$, $AB = \sqrt{2}AC$.
Hence

$$r_n + r_{n+1} = \sqrt{2}(r_n - r_{n+1})$$
$$(\sqrt{2}+1)r_{n+1} = (\sqrt{2}-1)r_n$$
$$r_{n+1} = \frac{\sqrt{2}-1}{\sqrt{2}+1}r_n$$

Then the sequence $\{r_n\}$ is a geometric progression with the first term $r_0 = a$ and common ratio $\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$.

$$r_n = a \cdot \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^n$$

(ii) The area of a circle C_n is

$$\begin{aligned} C_n &= \pi r_n^2 \\ &= \pi a^2 \cdot \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^{2n} \\ &= \pi a^2 \cdot \left(\frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)^2)}\right)^n \\ &= \pi a^2 \cdot \left(\frac{3-2\sqrt{2}}{3+2\sqrt{2}}\right)^n \\ &= \pi a^2 \cdot \left(\frac{3-2\sqrt{2}}{3+2\sqrt{2}}\right)^n \end{aligned}$$

Then the series $\{C_n\}$ is a geometric series with first term $C_0 = \pi a^2$ and common ratio $\dfrac{3-2\sqrt{2}}{3+2\sqrt{2}}$.
Since common ratio is between -1 and 1, the series converges.

$$\begin{aligned}
\sum_{n=0}^{\infty} C_n &= \frac{\pi a^2}{1-\frac{3-2\sqrt{2}}{3+2\sqrt{2}}} \\
&= \frac{\pi a^2}{\frac{(3+2\sqrt{2})-(3-2\sqrt{2})}{3+2\sqrt{2}}} \\
&= \frac{\pi a^2}{\frac{4\sqrt{2}}{3+2\sqrt{2}}} \\
&= \frac{\pi a^2(3+2\sqrt{2})}{4\sqrt{2}} \\
&= \frac{4+3\sqrt{2}}{4}\pi a^2
\end{aligned}$$

Exercise

[1] Find the siesr five terms of the following sequences whose n-th terms are:

(i) $a_n = 3n - 5$

(ii) $a_n = (-2)^{n-1}$

(iii) $a_n = (n+1)(2n+1)$

(iv) $a_n = \sin \dfrac{n\pi}{4}$

[2] Evaluate:

(i) $\displaystyle\sum_{r=0}^{4} r^3$

(ii) $\displaystyle\sum_{r=1}^{4} (r^2+1)$

(iii) $\displaystyle\sum_{r=2}^{5} r(r+1)$

(iv) $\displaystyle\sum_{r=4}^{6} (3r-1)$

[3] (i) Given that an arithmetic progression $a_1,\ a_2,\ \cdots, a_n,\ \cdots,$ which satisfies

$$a_1 + a_3 = 3, \quad \text{and} \quad a_2 + a_4 = 27$$

Express a_n in terms of n.

(ii) Given that a geometric progression $b_1,\ b_2,\ \cdots, b_n,\ \cdots,$ which satisfies

$$b_1 + b_3 = 3, \quad \text{and} \quad b_2 + b_4 = 27$$

Express b_n in terms of n.

[4] Let $\{a_n\}$ be an arithmetic progression, which satisfies:

$$a_{10} + a_{11} + a_{12} + a_{13} + a_{14} = 365$$

and

$$a_{15} + a_{17} + a_{19} = -6$$

(i) Find a_1 and the common difference d.

(ii) Let $S_n = a_1 + a_2 + \cdots + a_n$. Find the maximum value of S_n.

[5] Find the sum

(i)

$$2 \cdot (2n-1) + 4 \cdot (2n-3)/6 \cdot (2n-5) + \cdots + 2n \cdot 1$$

(ii)

$$\frac{1}{1} + \frac{1}{1+2} + + \frac{1}{1+2+3} + \cdots + + \frac{1}{1+2+3+\cdots+n}$$

[6] Given that a sequence $\{a_n\}$ such that

$$a_1 = 1, \quad \text{and} \quad a_{n+1} = 3a_n + 2 \quad (n = 1,\ 2,\ 3,\ \cdots)$$

(i) Express a_n in the terms of n.

(ii) Find the sum $a_1 + a_2 + \cdots + a_n$, in the terms of n.

[7] Given that a sequence $\{a_r\}$ such that

$$a_1 = 4, \quad \text{and} \quad a_{r+1} = 4a_r + 2^{r+1} \quad (r = 1,\ 2,\ 3,\ \cdots)$$

(i) Express a_n in the terms of n.

(ii) Find $\sum_{r=1}^{n} a_r$

[8] Let $S_n = a_1 + a_2 + \cdots + a_n$.
If$S_n = 3a_n + 2n - 1$, express a_n in the terms of n.

[9] Express a_n and b_n in the terms of n.

(i) $a_1 = \dfrac{1}{4}$ and $a_{n+1} = \dfrac{a_n}{3a_n + 1} \quad (n = 1,\ 2,\ 3,\ \cdots)$

(ii) $b_= 1$ and $b_{n+1} = 2b_n + 3^n \quad (n = 1,\ 2,\ 3,\ \cdots)$

[10] Given that two sequences $\{a_n\}$ and $\{b_n\}$, such that, for $n \geq 1$,

$$a_{n+1} = \frac{4a_n + b_n}{6}$$
$$b_{n+1} = \frac{-a_n + 2b_n}{6}$$

and

$$a_1 = 1, \quad \text{and} \quad b_1 = -2$$

(i) Show that $4a_{n+2} - 4a_{n+1} + a_n = 0$ for $n \geq 1$.

(ii) Show that the sequence $\{2^n a_n\}$ is an arithmetic progrssion.

(iii) Express a_n and b_n in the terms of n.

[11] Let c be a constant such that $c \neq \frac{4}{3}$.

Let $\{a_n\}$ be a sequence such that

$$\begin{cases} a_1 = 3, \quad \text{and} \quad a_2 = 7 \\ a_{n+2} - (3c-2)a_{n+1} + (3c-3)a_n = 0 \quad (n \geq 1) \end{cases} \qquad \cdots (*)$$

(i) Express (*) in the form

$$a_{n+1} - pa_{n+1} = q(a_{n+1} - pa_n) \quad (n \geq 1)$$

where p and q are constants to be found.

(ii) Express a_n in the terms of n.