Tohoku University

[1]

Assume that two tangents of the curve $y = \sin x$ are perpendicular to each other. Find the y-coordinate of the points of intersection of these tangents.

[2]

Let a ($\neq 1$) be a positive real number and let n be a positive integer.
Given that an inequality:

$$\log_a(x-n) > \frac{1}{2}\log_q(2n-x)$$

(1) When $n=6$, find all integers x satisfy the given inequality.

(2) Find the necessary and sufficient condition for n such that there exist integers x satisfying the given inequality.

[3]

Let a be a real number and let $\{x_n\}$ be a sequence defined by

$$x_1 = a, \quad x_{n+1} = x_n + x^2 \ (n = 1,\ 2,\ 3,\ \cdots)$$

(1) Show that the sequence is divergent, when $a > 0$.

(2) Find the limit of the sequence $\{x_n\}$, when $-1 < a < 0$.

[4]

Let $A(x)$ be a polynomial whose coefficients are real numbers. Let $[A(x)]$ be the remainder when $A(x)$ is divided by x^2+1.

(1) Find $[2x^2+x+3]$, $[x^5-1]$ and $[[2x^2+x+3][x^5-1]]$.

(2) Show that
$$[A(x)B(x)] = [[A(x)][B(x)]]$$
for any polynomials $A(x)$ and $B(x)$.

(3) Show that
$$[(x\sin\theta+\cos\theta)^2] = x\sin 2\theta + \cos 2\theta$$
for any real number θ.

(4) Find all couples (a, b) of real numbers such that
$$[(ax+b)^4] = -1$$

[5]

(1) Show that

$$\int_{-1}^{1} \frac{\sin^2(\pi x)}{1+e^x}\, dx = \int^{1} \sin^2(\pi x)\, dx = \frac{1}{2}$$

(2) Find a function $f(x)$ such that

$$(1+e^x)f(x) = \sin^2(\pi x) + \int_{-1}^{1} (e^x - e^t + 1) f(t)\, dt$$

[6]

In a bag there are 10 balls, 5 red and 5 white.
We put out randomly one ball from this bag and put one white ball in this bag.
When we repeat m times of this action, we let $p(m,k)$ be the probability that we took out k red balls from the bag.
Assume that $n \geq 2$.

(1) Express $p(n+1,2)$ with respect to $p(n,2)$ and $p(n,1)$.

(2) Find $p(n,1)$.

(3) Find $p(n,2)$.