## Tokyo University

[1]

Find the value of

$$\int_0^1 \left(x^2 + \frac{x}{\sqrt{1+x^2}}\right)\left(1 + \frac{x}{(1+x^2)\sqrt{1+x^2}}\right) dx$$

$$\begin{aligned}&\int_0^1 \left(x^2 + \frac{x}{\sqrt{1+x^2}}\right)\left(1 + \frac{x}{(1+x^2)\sqrt{1+x^2}}\right) dx \\ &\qquad = \int_0^1 \left(x^2 + \frac{x}{\sqrt{1+x^2}} + \frac{x^3}{(1+x^2)\sqrt{1+x^2}} + \frac{x^2}{(1+x^2)^2}\right) dx\end{aligned}$$

i) $\displaystyle\int_0^1 x^2\, dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$

ii) $\displaystyle\int_0^1 \left(\frac{x}{\sqrt{1+x^2}} + \frac{x^3}{(1+x^2)\sqrt{1+x^2}}\right) dx$

Substitute $u = 1 + x^2$,

$$\frac{du}{dx} = 2x, \qquad dx = \frac{du}{2x}$$

And

| $x$ | $0 \to 1$ |
|---|---|
| $u$ | $1 \to 2$ |

Then

$$\begin{aligned}\int_0^1 \left(\frac{x}{\sqrt{1+x^2}} + \frac{x^3}{(1+x^2)\sqrt{1+x^2}}\right) dx &= \int_1^2 \left(\frac{x}{\sqrt{u}} + \frac{x^3}{u\sqrt{u}}\right)\frac{du}{2x} \\ &= \frac{1}{2}\int_1^2 \left(\frac{1}{\sqrt{u}} + \frac{x^2}{u\sqrt{u}}\right) du \\ &= \frac{1}{2}\int_1^2 \left(\frac{1}{\sqrt{u}} + \frac{u-1}{u\sqrt{u}}\right) du \\ &= \frac{1}{2}\int_1^2 \left(2u^{-\frac{1}{2}} - u^{-\frac{3}{2}}\right) du \\ &= \frac{1}{2}\left[4\sqrt{u} + \frac{2}{\sqrt{u}}\right]_1^2 \\ &= 2\sqrt{2} + \frac{1}{\sqrt{2}} - 2 - 1 \\ &= \frac{5\sqrt{2} - 6}{2}\end{aligned}$$

iii) $\displaystyle\int_0^1 \frac{x^2}{(1+x^2)^2}\, dx$

Substitute $x = \tan\theta$,

$$\frac{dx}{d\theta} = \frac{1}{\cos^2\theta}, \qquad dx = \frac{d\theta}{\cos^2\theta}$$

And

$$\begin{array}{c|ccc} x & 0 & \to & 1 \\ \hline \theta & 0 & \to & \frac{\pi}{4} \end{array}$$

Then

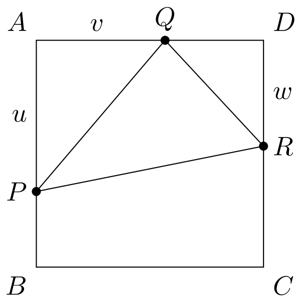
$$\begin{aligned}
\int_0^1 \frac{x^2}{(1+x^2)^2}\ dx &= \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{(1+\tan^2\theta)^2}\ \frac{d\theta}{\cos^2\theta} \\
&= \int_0^{\frac{\pi}{4}} \frac{\frac{\sin^2\theta}{\cos^2\theta}}{(\frac{1}{\cos^2\theta})^2}\ \frac{d\theta}{\cos^2\theta} \\
&= \int_0^{\frac{\pi}{4}} \sin^2\theta\ d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2}\ d\theta \\
&= \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{8} - \frac{1}{4}
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^1 \left(x^2 + \frac{x}{\sqrt{1+x^2}}\right)\left(1 + \frac{x}{(1+x^2)\sqrt{1+x^2}}\right)\ dx \\
&= \frac{1}{3} + \frac{5\sqrt{2}-6}{2} + \left(\frac{\pi}{8} - \frac{1}{4}\right) \\
&= \frac{\pi}{8} + \frac{5\sqrt{2}}{2} - \frac{35}{12}
\end{aligned}$$

[2]

Given that a square $ABCD$, whose side's length is 1. Let three points $P$, $Q$ and $R$ are on the sides $AB$, $AD$ and $CD$ respectively and the area of the triangle $APQ$ and the area of the triangle $PQR$ are both $\dfrac{1}{3}$.
Find the maximum and minimum values of $\dfrac{DR}{AQ}$.




Let $AP = u,\ AQ = v$ and $DR = w$, then $0 \leq u \leq 1,\ 0 \leq v \leq 1$ and $0 \leq w \leq 1$.
Since the area of the triangle $APQ$ and the area of the triangle $PQR$ are both $\dfrac{1}{3}$,

$$\triangle APQ = \frac{1}{2}uv = \frac{1}{3} \quad \cdots \quad (1)$$

$$\triangle PQR = \text{Trapeze}(APRD) - \triangle APQ - \triangle DQR = \frac{1}{2}(u+w)\cdot 1 - \frac{1}{2}uv - \frac{1}{2}(1-v)w = \frac{1}{3} \quad \cdots \quad (2)$$

From (1),

$$v = \frac{2}{3u}$$

Since $0 \leq v \leq 1$, $0 \leq \dfrac{2}{3u} \leq 1$.
Then $\dfrac{2}{3} \leq u \leq 1$.

From (2),

$$u - uv + vw = \frac{2}{3}$$
$$u - u\cdot\frac{2}{3u} + \frac{2}{3u}\cdot w = \frac{2}{3}$$
$$w = \frac{1}{2}(4u - 3u^2)$$

Then

$$\frac{DR}{AQ} = \frac{w}{v} = \frac{1}{2}(4u-3u^2)\cdot\frac{3u}{2} = \frac{3}{4}(4u^2 - 3u^3)$$

Let $f(u) = \dfrac{3}{4}(4u^2 - 3u^3)$,

$$f'(u) = \frac{3}{4}(8u - 9u^2) = \frac{3}{4}u(8 - 9u)$$

When $f'(u) = 0$, $u = 0,\ \dfrac{8}{9}$.
The variation table of $f$ is

| $u$ | $\frac{2}{3}$ | | $\frac{8}{9}$ | | $1$ |
|---|---|---|---|---|---|
| $f'(u)$ | | $+$ | $0$ | $-$ | |
| $f(u)$ | $\frac{2}{3}$ | $\nearrow$ | $\frac{64}{81}$ | $\searrow$ | $\frac{3}{4}$ |

Hence the maximum value of $\dfrac{DR}{AQ}$ is $\dfrac{64}{81}$

and the minimum value of $\dfrac{DR}{AQ}$ is $\dfrac{2}{3}$

[3]

Given that 5 points $A(2,0,0)$, $B(0,2,0)$, $C(-2,0,0)$, $D(0,-2,0)$ and $E(0,0,-2)$. Let $\alpha$ be a plane passing through the midpoint $M$ of $AB$ and the midpoint $N$ of $AD$ and parallel to the line $AE$. And given that a point $P(p,0,2)$, where $2 < p < 4$.

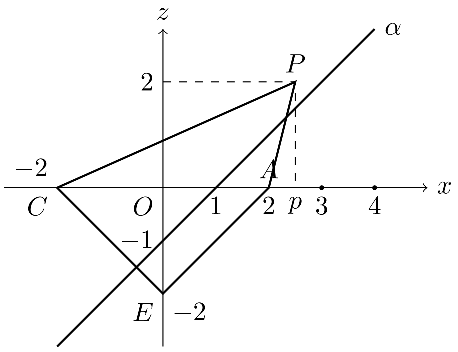
(1) Sketch the section of the octahedron $PABCDE$ by the plane whose equation is $y = 0$, and sketch the intersection of the two planes $\alpha$ and $y = 0$ on the same coordinate system.

(2) Find the range of value $p$ such that the section of the octahedron $PABCDE$ and the plane $\alpha$ forms an octagon.

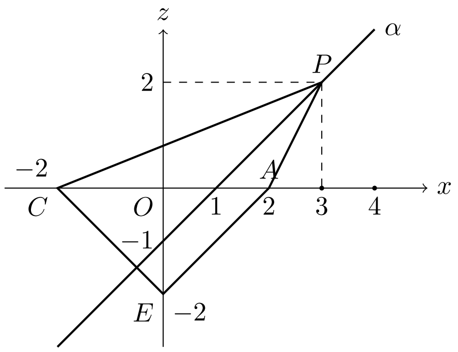
(3) Assume that the value of $p$ is in the range found in the question (2).
When a point $(x, y, z)$ is moving in the intersection of the octahedron $PABCDE$ and the plane $\alpha$ such that $y \geq 0$ and $z \geq 0$, find the area of the region which is formed by the moving points $(y, z)$.

(1)

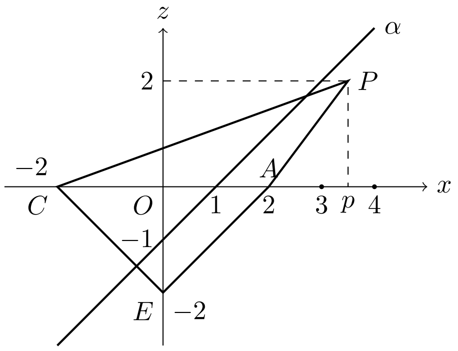
$2 < p < 3$




$p = 3$




$3 < p < 4$



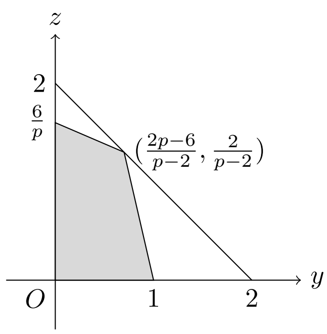
(2) When $2 = p \leq 3$, the plane $\alpha$ cuts the octahedron $PABCDE$ at point on the sides $CE,\ BE,\ DE,\ AB,\ AD$ and $PA$; six points. Then it is not formed an octagon.
When $3 < p < 4$, the plane $\alpha$ cuts the octahedron $PABCDE$ at point on the sides $CE,\ BE,\ DE,\ AB,\ AD,\ PB,\ PD$ and $PC$; eight points. The it forms an octagon.
Hence the required region of the value $p$ is

$$3 < p < 4$$

(3) We shall see the reflection on the $xz$-plane of the intersection of the octahedron $PABCDE$ and the plane $\alpha$, where $y \geq 0$ and $z \geq 0$, using a diagram in (2).
The equation of the line $PC$ is $z = \dfrac{2}{p+2}(x+2)$ and the equation of the line $\alpha$ is $z = x - 1$.
Then the coordinates of the intersection of the two lines are $(x, z) = (1 + \dfrac{6}{p}, \dfrac{6}{p})$.
Hence the coordinates of the intersection of the side $PC$ and the plane $\alpha$ are $(1 + \dfrac{6}{p}, 0, \dfrac{6}{p})$.
The equation of the line $PO$ is $z = \dfrac{2}{p}x$. Solving the simultaneous equations $z = \dfrac{2}{p}x$ and $z = x - 1$, we can find the coordinates of the intersection of $PO$ and the line $\alpha$ as $(x, z) = (\dfrac{p}{p-2}, \dfrac{2}{p-2})$.
A relation between the $y$ and $z$ coordinates of the side $PB$ is represented as $z = -y + 2$.
Then, when $z = \dfrac{2}{p-2}$, $y = \dfrac{2p-6}{p-2}$.
Hence the coordinates of the intersection of the side $PB$ and the plane $\alpha$ are $(\dfrac{p}{p-2}, \dfrac{2p-6}{p-2}, \dfrac{2}{p-2})$.

The coordinates of the intersection of the side $AB$ and the plane $\alpha$ are $M(1, 1, 0)$.

Therefore our required region which is formed by the points $(y, z)$ is as the diagram below:




Hence the area of the required region is

$$\begin{aligned}
\text{Area} &= \frac{1}{2} \cdot 2 \cdot 2 - \frac{1}{2}\left(2 - \frac{6}{p}\right) \cdot \frac{2p-6}{p-2} - \frac{1}{2}(2-1) \cdot \frac{2}{p-2} \\
&= 2 - \frac{2(p-3)^2}{p(p-2)} - \frac{1}{p-2} \\
&= \frac{2p(p-2) - 2(p-3)^2 - p}{p(p-2)} \\
&= \frac{2p^2 - 4p - 2(p^2 - 6p + 9) - p}{p(p-2)} \\
&= \frac{7p - 18}{p(p-2)}
\end{aligned}$$

[4]

Let $n$ be an integer which is larger than or equal to 1.

(1) Find the largest common divisor $d_n$ of $n^2+1$ and $5n^2+9$.

(2) Show that $(n^2+1)(5n^2+9)$ is not a square of any integer.

(1) Use the Euclidian Algorithm.
Since $5n^2+9=5(n^2+1)+4$, the largest common divisor of $n^2+1$ and $5n^2+9$ is the largest common divisor of $n^2+1$ and 4.

(i) When $n$ is even, we can write down as $n=2k$,

$$n^2+1=(2k)^2+1=4k^2+1$$

Then $n^2+1$ and 4 are prime each other.
Hence the largest common divisor of $n^2+1$ and 4 is 1.

(ii) When $n$ is odd, we can write down as $n=2k+1$,

$$n^2+1=(2k+1)^2+1=4k^2+4k+2=2(2k^2+2k+1)$$

Then the largest common divisor of $n^2+1$ and 4 is 2.

Hence the largest common divisor of $n^2+1$ and $5n^2+9$ is

$$d_n=\begin{cases}1 & \text{(when } n \text{ is even).}\\ 2 & \text{(when } n \text{ is odd).}\end{cases}$$

(2) (i) When $n$ is even,
the largest common divisor of $n^2+1$ and $5n^2+9$ is 1.
Then for $(n^2+1)(5n^2+9)$ is a square of some integer, each $n^2+1$ and $5n^2+9$ are a square of some integer.
As $n^2<n^2+1<(n+1)^2$, $n^2+1$ is not a square of any integer.
Hence $(n^2+1)(5n^2+9)$ is not a square of any integer.

(ii) When $n$ is odd,
the largest common divisor of $n^2+1$ and $5n^2+9$ is 2.
If $(n^2+1)(5n^2+9)$ is a square of some integer, we can write down as

$$n^2+1=2p^2 \quad \text{and} \quad 5n^2+9=2q^2$$

and $p$ and $q$ are prime each other.

$$(5n^2+9)-(n^2+1)=2q^2-2p^2$$
$$4n^2+8=2(q^2-p^2)$$
$$4(n^2+2)=2(q+p)(q-p)$$

Since $n$ is odd, $n^2+2$ is also odd, then $4(n^2+2)$ is divisible by 4 but not divisible by 8.
If $q+p$ is odd, $q-p$ is also odd, then $2(q+p)(q-p)$ is not divisible by 4.
If $q+p$ is even, $q-p$ is also even, then $2(q+p)(q-p)$ is divisible by 8.
Both case, $4(n^2+2)$ and $2(q+p)(q-p)$ are not equal.
Contradiction.
Hence $(n^2+1)(5n^2+9)$ is not a square of any integer.

[5]

(1) Let $n$ be an integer which is larger than or equal to 1. Show that the equation

$$x^{2n-1} = \cos x$$

has only one real root $a_n$.

(2) For $a_n$ at the part of question (1), show that $\cos a_n > \cos 1$.

(3) For the sequence $a_1,\ a_2,\ a_3,\ \cdots,\ a_n,\ \cdots$ defined by (1), find

$$a = \lim_{n\to\infty} a_n, \quad b = \lim_{n\to\infty} a_n^n, \quad c = \lim_{n\to\infty} \frac{a_n^n - b}{a_n - a}$$

(1) i) When $x \leq -\frac{\pi}{2}$,

$$x^{2n-1} \leq -\left(\frac{\pi}{2}\right)^{2n-1} < -1$$

Since $-1 \leq \cos x$, we have

$$x^{2n-1} < \cos x$$

ii) When $-\frac{\pi}{2} < x \leq 0$,

$$x^{2n-1} \leq 0, \quad \text{and} \quad 0 < \cos x \leq 1$$

Then

$$x^{2n-1} < \cos x$$

iii) When $0 < x < \frac{\pi}{2}$,
let $f(x) = x^{2n-1} - \cos x$, then

$$f'(x) = (2n-1)x^{2n-2} + \sin x > 0$$

Therefore the function $f(x)$ is strictly increasing in the interval $0 < x < \frac{\pi}{2}$.

$$f(0) = -1 < 0, \quad \text{and} \quad f(\frac{\pi}{2}) = \left(\frac{\pi}{2}\right)^{2n-1} > 0$$

Hence there exist one and only one value $x = a_n$ such that $f(a_n) = 0$ and $0 < a_n < \frac{\pi}{2}$.

iv) When $\frac{\pi}{2} \leq x$,

$$x^{2n-1} \geq \left(\frac{\pi}{2}\right)^{2n-1} > 1$$

and $\cos x \leq 1$.
Then

$$x^{2n-1} > \cos x$$

Hence the equation $x^{2n-1} = \cos x$ has one and only one root $a_n$ such that $0 < a_n < \frac{\pi}{2}$.

(2) Since $f(1) = 1 - \cos 1 > 0$ and $f(0) = -1 < 0$,
then

$$0 < a_n < 1$$

Since $\cos x$ is strictly decreasing in the interval $0 < x < 1$,

$$\cos a_n > \cos 1$$

(3) From $a_n^{2n-1} = \cos a_n$,

$$a_n = (\cos a_n)^{\frac{1}{2n-1}}$$

Since $\cos a_n > \cos 1$,

$$(\cos 1)^{\frac{1}{2n-1}} < (\cos a_n)^{\frac{1}{2n-1}} = a_n < 1$$

$$\lim_{n\to\infty} (\cos 1)^{\frac{1}{2n-1}} \leq \lim_{n\to\infty} a_n \leq 1$$

$$1 \leq \lim_{n\to\infty} a_n \leq 1$$

Hence

$$a = \lim_{n\to\infty} a_n = 1$$

$$(a_n^n)^2 = a_n^{2n} = a_n a_n^{2n-1} = a_n \cos a_n$$

Then

$$a_n^n = \sqrt{a_n \cos a_n}$$

Hence

$$b = \lim_{n\to\infty} a_n^n = \lim_{n\to\infty} \sqrt{a_n \cos a_n} = \sqrt{\cos 1}$$

$$\frac{a_n^n - b}{a_n - a} = \frac{\sqrt{a_n \cos a_n} - \sqrt{\cos 1}}{a_n - 1}$$

Let $g(x) = \sqrt{x \cos x}$,
then

$$g'(x) = \frac{\cos x - x \sin x}{2\sqrt{x \cos x}}$$

From the mean value theorem,
there exist a real number $\alpha$ such that $a_n < \alpha < 1$ and

$$\frac{g(a_n) - g(1)}{a_n - 1} = g'(\alpha)$$

then

$$\frac{a_n^n - b}{a_n - a} = g'(\alpha)$$

Since $a_n < \alpha < 1$ and $\lim_{n\to\infty} a_n = 1$,

$$\lim_{n\to\infty} \alpha = 1$$

Hence

$$\begin{aligned} c &= \lim_{n\to\infty} \frac{a_n^n - b}{a_n - a} \\ &= \lim_{n\to\infty} g'(\alpha) \\ &= g'(1) \\ &= \frac{\cos 1 - \sin 1}{2\sqrt{\cos 1}} \end{aligned}$$

[6]

Given that complex numbers $\alpha,\ \beta, \gamma$ and $\delta$ and real numbers $a$ and $b$ which satisfy the three conditions:

(i) $\alpha,\ \beta, \gamma$ and $\delta$ are different number each other.

(ii) $\alpha,\ \beta, \gamma$ and $\delta$ are the roots of quartic equation $x^4 - 2x^3 - 2ax + b = 0$.

(iii) The real part of a complex number $\alpha\beta + \gamma\delta$ is 0 and the imaginary part of $\alpha\beta + \gamma\delta$ is not 0.

Answer the following qustions.

(1) Show that two of the complex numbers $\alpha,\ \beta, \gamma$ and $\delta$ are real, and another two are conjugate complex numbers each other.

(2) Express $b$ with terms of $a$.

(3) Sketch the loci of the point $\alpha + \beta$ on the Argand diagram.

(1) Since the all of the coefficients of the quartic equation $x^4 - 2x^3 - 2ax + b = 0$ are real, if $u + iv$ is a root of this equation, its conjugate $u - iv$ is also a root of this equation.
Then the roots of the equation $x^4 - 2x^3 - 2ax + b = 0$ are all real number, or two real numbers and two complex numbers ( conjugate each other), or four non-real numbers.

a) If all roots of the equation are real, the condition (ii) is not satisfied.

b) If all roots are non-real numbers, we can write down the four roots as

$$s + it,\ s - it,\ u + iv,\ u - iv \qquad \text{where } t \neq 0 \text{ and } v \neq 0)$$

Assume that $\alpha = s + it,\ \beta = s - it,\ \gamma = u + iv$ and $\delta = u - iv$, then

$$\alpha\beta + \gamma\delta = (s + it)(s - it) + (u + iv)(u - iv) = s^2 + t^2 + u^2 + v^2$$

which is not satisfied the condition (iii).
Assume that $\alpha = s + it,\ \beta = u + iv,\ \gamma = s - it$ and $\delta = u - iv$, then

$$\alpha\beta + \gamma\delta = (s + it)(u + iv) + (s - it)(u - iv) = 2su - 2tv$$

which is not satisfied the condition (iii).

Hence two of the roots of the equation $x^4 - 2x^3 - 2ax + b = 0$ are real numbers, and another tow roots are complex numbers which are conjugate each other.

(2) Let $p,\ q,\ u + iv$ and $u - iv$ be the roots of the quartic equation $x^4 - 2x^3 - 2ax + b = 0$.

Assume that $\alpha = p,\ \beta = q,\ \gamma = u + iv$ and $\delta = u - iv$.

Then

$$\alpha\beta + \gamma\delta = p + q + (u + iv) + (u - iv) = p + q + 2u$$

which is not satisfied the condition (iii).

Assume that $\alpha = p,\ \beta = u + iv,\ \gamma = q$ and $\delta = u - iv$.
Then

$$\alpha\beta + \gamma\delta = p(u + iv) + q(u - iv) = u(p + q) + iv(p - q)$$

From the condition (iii),

$$u(p+q)=0, \quad \text{and} \quad v(p-q)\neq 0$$

Then $u=0$ or $q=-p$ and $v\neq 0$ and $p\neq q$.

Since $\alpha,\ \beta,\ \gamma$ and $\delta$ are the roots of the quartic equation

$$x^4-2x^3-2ax+b=0$$

We have the formula

$$\begin{cases} \alpha+\beta+\gamma+\delta=2 \\ \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=0 \\ \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=2a \\ \alpha\beta\gamma\delta=b \end{cases}$$

(a) When $u=0$, we can write down the roots as

$$\alpha=p,\ \beta=iv,\ \gamma=q,\ \delta=-iv$$

Then

$$\begin{cases} p+iv+q-iv=2 \\ ipv+pq-ipv+iqv+v^2-iqv=0 \\ ipqv+pv^2-ipqv+qv^2=2a \\ pqv^2=b \end{cases}$$

$$\begin{cases} p+q=2 \\ pq+v^2=0 \\ v^2(p+q)=2a \\ pqv^2=b \end{cases}$$

Therefore $pq=-v^2,\ a=v^2$ and $b=pqv^2$.
Hence

$$b=-a^2$$

(b) When $q=-p$, we can write down the roots as

$$\alpha=p,\ \beta=u+iv,\ \gamma=-p,\ \delta=u-iv$$

Then

$$\begin{cases} p+(u+iv)-p+(u-iv)=2 \\ p(u+iv)-p^2+p(u-iv)-p(u+iv)+(u+iv)(u-iv)-p(u-iv)=0 \\ -p^2(u+iv)+p(u+iv)(u-iv)-p^2(u-iv)-p(u+iv)(u-iv)=2a \\ -p^2(u+iv)(u-iv)=b \end{cases}$$

$$\begin{cases} 2u=2 \\ -p^2+u^2+v^2=0 \\ -2p^2u=2a \\ -p^2(u^2+v^2)=b \end{cases}$$

Therefore $u=1,\ p^2=u^2+v^2,\ a=-p^2$ and $b=-p^2(u^2+v^2)$.
Hence

$$b=-a^2$$

Hence

$$b=-a^2$$

(3) Form the answer of the question (2), we can say that $u = 0$ or $q = -p$.

(a) When $u = 0$,

$$\alpha = p,\ \beta = iv,\ \gamma = q,\ \delta = -iv$$

Then

$$\alpha + \beta = p + iv$$

Let $\alpha + \beta = x + iy$, then $x = p$ and $y = v$.
Since $p + q = 2$ and $pq + v^2 = 0$, $p(2 - p) + v^2 = 0$.

$$x(2 - x) + y^2 = 0$$

$$(x - 1)^2 - y^2 = 1$$

and $y \neq 0$ ( since $v \neq 0$).

(b) When $q = -p$,

$$\alpha = p,\ \beta = u + iv,\ \gamma = -p,\ \delta = u - iv$$

Then

$$\alpha + \beta = p + (u + iv) = (p + u) + iv$$

Let $\alpha + \beta = x + iy$, then $x = p + u$ and $y = v$.
Since $u = 1,\ p^2 = u^2 + v^2$,

$$(x - 1)^2 = 1 + y^2$$

$$(x - 1)^2 - y^2 = 1$$

and $y \neq 0$.

Hence the equation of the loci of $\alpha + \beta$ is

$$(x - 1)^2 - y^2 = 1$$