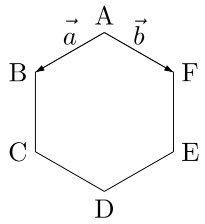
Vector

Example 1

Given a regular hexagon $ABCDEF$ and let $\vec{a} = \overrightarrow{\mathrm{AB}}$, $\vec{b} = \overrightarrow{\mathrm{AF}}$.
Present the following vectors with $\vec{a}$ and $\vec{b}$.



(1) $\overrightarrow{\mathrm{AC}}$ (2) $\overrightarrow{\mathrm{AD}}$
(3) $\overrightarrow{\mathrm{CD}}$ (4) $\overrightarrow{\mathrm{CE}}$

[1] Given a parallelogram $ABCD$, and let intersection of two diagonals be O. Let E be the point dividing internally the segmentt AB in the ratio $1:2$, and let F be the midpoint of BC.
Present the following vectors by $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AD}}$.

(1) $\overrightarrow{\mathrm{AO}}$ (2) $\overrightarrow{\mathrm{DO}}$
(3) $\overrightarrow{\mathrm{AF}}$ (4) $\overrightarrow{\mathrm{EF}}$

[2] Present the following $\vec{x}$ and $\vec{y}$ by $\vec{a}$ and $\vec{b}$.

(1) $\vec{x} + 2(\vec{x} - \vec{b}) = 4(\vec{b} - 2\vec{a}) - \vec{a}$

(2) $\frac{1}{2}(\vec{b} + \vec{x}) + 3(\vec{x} - \frac{1}{2}\vec{b}) = \vec{0}$

(3) $2\vec{x} + 3\vec{y} = \vec{a} + \vec{b}, \quad 3\vec{x} + 2\vec{y} = \vec{b}$

Example 2

(1) Let $\vec{a} = (-1, 2),\ \vec{b} = (1, -2),\ \vec{c} = (4, 4)$. Find the real numbers $l,\ k$, which satisfies $\vec{c} = l\vec{a} + k\vec{b}$.

(2) Find the unit vector who has the same direction as $\vec{a} = (3, 5)$.

[3] Let $\vec{a} = (-1, 4),\ \vec{b} = (3, 2),\ \vec{c} = (0, -5)$.
Find the following vecors.

(1) $-3\vec{a} + \vec{b}$

(2) $\vec{a} - 2\vec{b} + 3\vec{c}$

(3) $6(\vec{c} - 2\vec{a}) - 5(-3\vec{b} + \vec{a})$

[4] (1) Let $\vec{a} = (1, 1),\ \vec{b} = (-1, -3),\ \vec{c} = (-3, -5)$. Find the real numbers $l,\ k$, which satisfies $\vec{c} = l\vec{a} + k\vec{b}$.

(2) Given a parallelogram $ABCD$, and let $A(1, 2),\ B(3, 5),\ C(-4, 0)$. Find the coordinate of the point D.

[5] Let $\vec{a} = (3, 1),\ \vec{b} = (1, 2),\ \vec{c} = \vec{a} + t\vec{b}$, where t is a real number.

(1) If $||\vec{c}|| = \sqrt{15}$, find the value of t.

(2) Find the minimal value of $||\vec{c}||$ and t at this time.

[6] Given $A,\ B,\ C$ be three points, which are not on a line. Let $\overrightarrow{AB} = \vec{a},\ \overrightarrow{AC} = \vec{b}$. Present the vector, by $\vec{a},\ \vec{b}$, whose direction is the bisection of $\angle ABC$.

Example 3

(1) Let $\vec{a} = (x_1, y_1),\ \vec{b} = (x_2, y_2)$ and the angle between $\vec{a}$ and $\vec{b}$ be θ, then prove that

$$||\vec{a}||||\vec{b}|| \cos\theta = x_1x_2 + y_1y_2$$

(2) Given two vectors $\vec{a} = (1, -1),\ \vec{b} = (2, x)$. Find the x, which satisfies the following condition.

(1) $\vec{a} \perp \vec{b}$ (2) The angle between $\vec{a}$ and $\vec{b}$ is $120°$ (3) $\vec{a} \parallel \vec{b}$

[7] Let $||\vec{a}|| = 4,\ ||\vec{b}|| = 5$ and the angle between $\vec{a}$ and $\vec{b}$ be as follows. Find the dot product of $\vec{a}$ and $\vec{b}$.

(1) $45°$ (2) $120°$ (3) $90°$ (4) $180°$

[8] Find the dot product of $\vec{a},\ \vec{b}$.

(1) $\vec{a} = (2, 0),\ \vec{b} = (2, 1)$
(2) $\vec{a} = (1, -1),\ \vec{b} = (3, 2)$
(3) $\vec{a} = (k + 2, k - 1),\ \vec{b} = (2k - 4, -k + 1)$
(4) $\vec{a} = (p + q, q),\ \vec{b} = (p - q, p)$

[9] Find the angle between $\vec{a},\ \vec{b}$.

(1) $\vec{a} = (1, \sqrt{3}),\ \vec{b} = (2, 0)$
(2) $\vec{a} = (3, 7),\ \vec{b} = (2, -5)$
(3) $\vec{a} = (-\sqrt{3} - 1, \sqrt{3} - 1),\ \vec{b} = (1, 1)$
(4) $\vec{a} = (\sqrt{2}, \sqrt{2}),\ \vec{b} = (\sqrt{3} - 1, \sqrt{3} + 1)$

[10] (1) Given $||\vec{a}|| = 1,\ ||\vec{b}|| = 5,\ ||2\vec{a} + \vec{b}|| = 3$. Find $\vec{a} \cdot \vec{b}$.

(2) Let $\vec{a} = (3, 7),\ \vec{b} = (2, -5)$. Find the minimal value of $||\vec{a} + t\vec{b}||$ and the value of t at this time.

Example 4

Given $\triangle ABC$, let D be the point dividing internally AB in the ratio $3:1$, E be the point dividing internally AC in the ratio $4:3$. And let P be the intersection of BE and CD, Q be the intersection of AP and BC.

(1) Present $\overrightarrow{BE}$, $\overrightarrow{CD}$ by $\overrightarrow{AB}$, $\overrightarrow{AC}$.

(2) Find the ratio $AP:PQ$.

[11] Given $\triangle OAB$ and let C be the point dividing internally OA in the ratio $5:2$, D be the point dividing internally OB in the ratio $3:4$ and M be the middle point of CD.

(1) Present $\overrightarrow{OM}$ by $\overrightarrow{OA}$, $\overrightarrow{OB}$.

(2) Let N be the intersection of OM and AB, find the ratio $ON:OM$ and $AN:NB$.

[12] Given a regular pentagon $ABCDE$ whose side's length is 1. Let $\overrightarrow{AB}=\vec{a}$, $\overrightarrow{BC}=\vec{b}$. Present $\overrightarrow{CD}$ by $\vec{a}$, $\vec{b}$.

[13] Given $\triangle ABC$ and let $\overrightarrow{AB}=\vec{c}$, $\overrightarrow{BC}=\vec{a}$, $\overrightarrow{CA}=\vec{a}$, and let $AB=c$, $BC=a$, $CA=b$.

(1) Let G be the centre of gravity of $\triangle ABC$. Present $\overrightarrow{AG}$ by $\vec{b}$ and $\vec{c}$.

(2) Present $||\overrightarrow{AG}||^2$ by a, b, c.

(3) Find the angle $\angle AGB$, when $a=\sqrt{12}$, $b=\sqrt{21}$, $c=\sqrt{3}$.

Example 5

[1] Find the equation of following lines.

(1) Line passing through the point $(-3, 4)$ and parallel with the vector $\vec{v} = (1, -2)$.

(2) Line passing through two points $(1, -3)$, $(-2, 4)$.

(3) Line passing tthrough he point $(5, -4)$ and perpendicular with the vector $\vec{v} = (1, -2)$.

[2] Find the equation of the circle, whose centre is $(2, 0)$ and passes through the point $(-1, 3)$.

[14] Find the equation of following lines.

(1) Line passing through the point $(\ 2\ , 4)$ and perpendicular with $\vec{n} = (1, -2)$.

(2) The perpendicular bisector of the segment whose vertexes are $(1, -3)$, $(-2, 4)$.

[15] Find the equation of the circle whose centre is $(-2, 3)$ and tangent to the line $x - y - 1 = 0$.

[16] Let O be the circumcentre of $\triangle ABC$ and let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$.

(1) Let A' be the intersection of BC and the line which passes through the point A and perpendicular to BC. Find the equation of the line AA'.

(2) Let H be the orthocentre of $\triangle ABC$. Present $\overrightarrow{OH}$ by $\vec{a}$, $\vec{b}$, $\vec{c}$.

Example 6

Given a tetrahedron $OABC$ and let D be the internally dividing point of AB in the ratio $1:2$, E be the internally dividing point of CD in the ratio $3:5$ and F be the internally dividing point of OE in the ratio $1:3$. And let G be the intersection of the line AF and the plane OBC. Let $\overrightarrow{OA}=\vec{a},\ \overrightarrow{OB}=\vec{b},\ \overrightarrow{OC}=\vec{c}$.

(1) Present $\overrightarrow{OE}$ by $\vec{a},\ \vec{b},\ \vec{c}$.

(2) Find the ratio $AG:FG$.

[17] Let $\vec{a},\ \vec{b},\ \vec{c},\ \vec{d}$ be the positional vectors of each vertex of a tetrahedron $ABCD$. Find the positional vector of the point which divides internally the segment between A and the centre of gravity of the triangle $\triangle BCD$ in the ratio $3:1$.

[18] Given a parallelogram $ABCD$ and let $A(1,-2,3),\ B(3,2,1),\ C(6,4,4)$.

(1) Find the coordinate of the point D.

(2) If $E(1,y,15)$ is on the plane ABC, find y.

[19] Given a regular tetrahedron $PABC$, whose side's length is 1, and let H be the intersection of the line passing through A and perpendicular to PBC and to the plane PBC. Let $\overrightarrow{PA}=\vec{a},\ \overrightarrow{PB}=\vec{b},\ \overrightarrow{PC}=\vec{c}$.

(1) Find the dot products $\vec{a}\cdot\vec{b},\ \vec{b}\cdot\vec{c},\ \vec{c}\cdot\vec{a}$.

(2) Present $\overrightarrow{PH}$ by $\vec{b},\ \vec{c}$.

(3) Evaluate the volume of the tetrahedron $PABC$.

Example 7

[1] Find the equation of following lines.

(1) Line passing through the point $(-3, 4, 5)$ and parallel to the vector $\vec{v} = (1, -2, 1)$.

(2) Line passing through two points $(1, -3, 7),\ (-2, 4, -2)$.

[2] Find the equation of following planes.

(1) Surface passing through the point $(-3, 4, 5)$ and perpendicular to the vector $\vec{v} = (1, -2, 1)$.

(2) Surface passing through three points $(1, -3, 7),\ (-2, 4, -2),\ (0, 2, 1)$.

[20] Find the equation of following lines.

(1) Line passing through the point $(1, 2, 3)$ and parallel to the vector $\vec{v} = (2, 3, 4)$.

(2) Line passing through two points $(0, 1, 3),\ (1, -2, 4)$.

[21] Find the equation of following planes.

(1) Surface passing through the point $(1, 2, 3)$ and perpendicular to the vector $\vec{v} = (7, 6, 5)$.

(2) Surface passing through three points $(-1, 2, 4),\ (2, 1, 6),\ (-3, 1, -2)$.

Example 8

[1] Find the intersection point of the following line and plane.

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z+2}{-2}, \quad 3x + 2y + z = 1$$

[2] Find the distance between the following point and the plane.

$$(1, -1, 2), \quad 2x - y + z - 1 = 0$$

[22] Find the intersection points of the following lines and the planes.

(1) $\dfrac{x+2}{-2} = y - 3 = \dfrac{y-1}{-3}, \quad 2x - y + 3z + 2 = 0$

(2) $\dfrac{x+1}{-2} = \dfrac{y-1}{3} = \dfrac{z+2}{-2}, \quad 2x - 3y + z - 1 = 0$

[23] Find the distance between the folioing points and the planes.

(1) $(2, 0, -1), \quad x + y - 2z + 3 = 0$

(2) $(-1, 1, -1), \quad 3x - y + 2z + 1 = 0$

[24] Find the equation of lines passing the point $(1, 2, 3)$ and perpendicular with the following planes.

(1) $2x + 3y + z + 1 = 0$

(2) $-x + 2y - 2z - 3 = 0$

Example 9

Prove the following statements.

(1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(2) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

(3) $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$

(4) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \times \vec{c})\vec{a}$

(5) $(\vec{a} \times \vec{b}) \perp \vec{a}$ and $(\vec{a} \times \vec{b}) \perp \vec{b}$

(6) $(\vec{a} \times \vec{b})^2 = (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2$

[25] Find the cross product $\vec{a} \times \vec{b}$.

(1) $\vec{a} = (1, -1, 1), \quad \vec{b} = (-2, 3, 1)$

(2) $\vec{a} = (-1, 1, 2), \quad \vec{b} = (1, 0, -1)$

[26] Find the unit vector which is perpendicular to the both vectors $\vec{a} = (1, 1, 3)$, $\vec{b} = (-3, 1, 4)$.

[27] Given two vectors $\vec{a} = (1, 0, -2)$, $\vec{b} = (-2, k, 4)$. Find k, if $\vec{a} \times \vec{b} = \vec{0}$

[28] Prove that the volume of parallelepiped whose three edges are represented by $\vec{a}$, $\vec{b}$, $\vec{c}$ is $||\vec{c} \cdot (\vec{a} \times \vec{b})||$.

Exercises

[1] Given a quadrilateral $OABC$. Let P be the centre of gravity of the triangle $\triangle OAB$ and Q be the centre of gravity of the triangle $\triangle OBC$. Let $\overrightarrow{\mathrm{OA}} = \vec{a}$, $\overrightarrow{\mathrm{OB}} = \vec{b}$, $\overrightarrow{\mathrm{OC}} = \vec{c}$.

(1) Find z if $\overrightarrow{\mathrm{PQ}} = x\vec{a} + y\vec{b} + z\vec{c}$

(2) if $\angle\mathrm{AOC} = 60°$, $OA = 3$, $OC = 2$, find $||\overrightarrow{\mathrm{PQ}}||$.

[2] Given a triangle $\triangle\mathrm{OAB}$ and let $OA = 2$, $OB = 3$, $\angle\mathrm{AOB} = 120°$, and M be the midpoint of AB and N be the midpoint of AM. Let $\overrightarrow{\mathrm{OA}} = \overrightarrow{a}$, $\overrightarrow{\mathrm{OB}} = \overrightarrow{b}$

(1) Find the dot product $\overrightarrow{a} \cdot \overrightarrow{b}$.

(2) Present $\overrightarrow{\mathrm{OM}}$ and $\overrightarrow{\mathrm{ON}}$ by $\overrightarrow{a}$, $\overrightarrow{b}$.

(3) Find the dot product $\overrightarrow{\mathrm{OM}} \cdot \overrightarrow{\mathrm{ON}}$.

(4) Let $\angle\mathrm{MON} = \theta$, evaluate $\cos\theta$.

[3] Given two vectors $\overrightarrow{a} = (1, x)$, $\overrightarrow{b} = (2, -1)$.

(1) If $\overrightarrow{a} + \overrightarrow{b}$ and $2\overrightarrow{a} - 3\overrightarrow{b}$ are perpendicular, find x.

(2) If $\overrightarrow{a} + \overrightarrow{b}$ and $2\overrightarrow{a} - 3\overrightarrow{b}$ are parallel, find x.

[4] Let $ABCD$ be a quadrilateral such as $AB \mathbin{/\!/} DC$, $AB = 6$, $CD = 4$. Let M be the midpoint of AB and N be the midpoint of AD. Let P be the point on the segment MN such as $MP : PN = 1 : 3$. And let Q be the intersection of CP and AB. Let $\overrightarrow{\mathrm{AB}} = \overrightarrow{a}$, $\overrightarrow{\mathrm{AD}} = \overrightarrow{b}$

(1) Present $\overrightarrow{\mathrm{AC}}$ and $\overrightarrow{\mathrm{AP}}$ by $\vec{a}, \vec{b}$.

(2) Find the ratio $CP : PQ$.

(3) If $AD = 5$, $\angle\mathrm{BAD} = 60°$, evaluate the length of CQ.

[5] Given three vectors $\vec{a} = (\cos\alpha,\ \sin\alpha,\ 0)$, $\vec{b} = (\sin\alpha,\ -\cos\alpha,\ t)$, $\vec{c} = (\sin\alpha,\ \cos\alpha,\ 0)$, where α, t are real numbers. Let $\vec{v} \neq \vec{0}$ be perpendicular to $\vec{a}$ and $\vec{b}$. Find $\cos\theta$, where θ is the angle between $\vec{v}$ and $\vec{c}$.

[6] Let $A(1,0,0), B(2,1,0), C(3,4,1)$, and $\overrightarrow{\mathrm{OA}} = \overrightarrow{a}$, $\overrightarrow{\mathrm{OB}} = \overrightarrow{b}$ $\overrightarrow{\mathrm{OC}} = \overrightarrow{c}$, and let α be the plane passing three points A, B, C.

(a) Let $P(x,y,z)$ be a point such that $\overrightarrow{\mathrm{OP}} = r\overrightarrow{a} + s\overrightarrow{b} + t\overrightarrow{c}$. Present $r,\ s,\ t$ using by $x, ; y, ; z$.

(b) When the point P is on the plane α, find the equation of $x,\ y,\ z$.

(c) Given the point $\mathrm{D}(4,5,7)$. Find the H on the plane α, which satisfies the condition $\mathrm{DH} \perp \mathrm{AB}$, $\mathrm{DH} \perp \mathrm{BC}$.

[7] Let α be the plane passing through three points $O(0,0,0), A(1,1,0), B(1,0,1)$.

(1) Find the unit vector parallel to the plane α and perpendicular to $\overrightarrow{\mathrm{OA}}$.

(2) Find the centre and radius of circumscribed circle of the triangle $\triangle\mathrm{OAB}$ on the plane α.